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Permanents of Multidimensional Matrices: Properties and Applications

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Abstract—The permanent of a multidimensional matrix is the sum of the products of entries over all diagonals. In this survey, we consider the basic properties of the multidimensional permanent, sufficient conditions for its positivity, available upper bounds, and the specifics of the permanents of polystochastic matrices. We prove that the number of various combinatorial objects can be expressed via multidimensional permanents. Special attention is paid to the number of 1-factors of uniform hypergraphs and the number of transversals in Latin hypercubes.

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INTRODUCTION

The notion of permanent of a two-dimensional matrix was presumably introduced by Cauchy in 1812 in [1], and so the history of permanents lasts for more than 200 years. The properties of permanents are well studied, and various applications were found for them in graph theory, in the theory of combinatorial structures, and even in physics. The most familiar use of permanents was the calculation of the number of perfect matchings in a bipartite graph. The study of permanents is the topic of many articles, and a significant part of them is devoted to the permanents of doubly stochastic matrices. The most complete description of the properties of permanents has been given in [2].

Various authors attempted repeatedly to generalize the notion of permanent to multidimensional matrices but no systematic study of the multidimensional permanent or its applications was carried out. As a consequence, there is no system of unified notations and terminology in this area. Here, we attempt at exposing the available properties of multidimensional permanents, describing possible ways of applying permanents to various problems of discrete mathematics, and giving the main problems in the theory of multidimensional permanents.

One of the first generalizations of the concept of permanent to the multidimensional case was proposed by Cayley in 1849 [3]. Then the permanent of a multidimensional matrix and its properties were mentioned by Muir in [4], where the multidimensional permanent appears as one of the possible generalizations of the determinant. By similar considerations, multidimensional permanents arise in books by Sokolov [5, 6], where several simple properties are given alongside definitions.

Until recently, the articles [7, 8] by Dow and Gibson of the 1980s were the only ones in which the multidimensional permanent is the main object under study. In these articles, some simple but important properties of the permanents of multidimensional matrices are given. In particular, the fact is noticed that the number of 1-factors of a simple d-partite uniform hypergraph coincides with the permanent of the d-dimensional adjacency matrix of its parts.

Recently some applications have been revealed for the permanents of multidimensional matrices that helped in solving several combinatorial problems. For example, using the multidimensional permanents, the author of this article substantially improved the upper bound for the number of transversals in Latin squares [9], while Linial and Luria obtained an upper bound for the number of Latin hypercubes [10].

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Apparently, the set of possible applications of the multidimensional permanents is substantially richer than the set of applications of the permanents of two-dimensional matrices. The multidimensional permanents are closely connected with the objects such as transversals in Latin hypercubes, 1-factors of hypergraphs, tilings of transitive graphs, MDS codes, and combinatorial designs. Many results on these objects can be reformulated in terms of the permanents of multidimensional matrices, and the multidimensional permanents in turn can help in solving some problems for these objects. As in the case of the two-dimensional matrices, the multidimensional permanents find their application in physics for describing the interaction of the elementary particles, which is exposed in [11].

The matrices whose permanents arise in various applications often have special structure. Usually these are nonnegative matrices whose nonzero entries are distributed regularly over planes. Therefore, it is actual to separately consider the permanents of multidimensional matrices for which the sums of the entries over planes of some dimension are fixed. These classes of matrices are in a sense a generalization of doubly stochastic matrices to the multidimensional case, and this article contains a number of interesting properties for one of such classes, the class of polystochastic matrices.

It is natural to suppose that the problems of computation and check of the positivity of the permanent of a multidimensional matrix are not easier than the analogous two-dimensional problems. As is known, the problem of computing the permanent of a two-dimensional $(0, 1)$-matrix lies in the class of $\#P$-complete problems; i.e., it is one of the most difficult computational problems. But for checking the positivity of the two-dimensional permanent, there is a simple polynomial algorithm based on the König–Frobenius algorithm. In the multidimensional case, already the recognition problem of the positivity of permanents of three-dimensional $(0, 1)$-matrices is NP-complete since it is equivalent to the NP-complete problem on the maximal 1-factor in a 3-uniform 3-partite hypergraph. Nevertheless, for some multidimensional matrices, polynomial algorithms for computing permanents were proposed [12].

In this article, alongside a review of the available results, we also expose some new results; most important among those are the construction of multidimensional matrices whose permanents express the number of some combinatorial objects, the construction of matrices showing that the minimum of the permanent on polystochastic matrices is not attained at a uniform matrix, a theorem on the positivity of the permanents of all polystochastic matrices of order 3, a theorem on the evenness of the number of transversals in Latin hypercubes of even order, as well as bounds for and the exact values of the number of transversals in some Latin hypercubes.

Let us briefly describe the structure of the article:

In Section 1, we define the main notions such as a multidimensional matrix, planes of a multidimensional matrix, and their directions, the permanent of a mutlidimensional matrix, and nonnegative and polystochastic matrices. We also consider the basic properties of the multidimensional permanents and calculate the permanents of several important examples of matrices.

In Section 2, we study the relationship between the multidimensional permanent and the number of various combinatorial objects. We give constructions of multidimensional matrices whose permanents make it possible to estimate the number of transversals in Latin hypercubes, the numbers of MDS codes and Latin hypercubes, the numbers of Steiner designs, A-designs, and H-designs, the number of tilings of transitive graphs and abelian groups, the number of perfect codes in the Boolean cube, and the number of Birch partitions. Many of these constructions are consequences of the fact that the number of 1-factors of every d-partite hypergraph is equal to the permanent of its d-dimensional adjacency matrix of parts. Also in Section 2, we point out the relationship between the permanent of the adjacency matrix of an arbitrary uniform hypergraph and the number of 1-factors of it and also give some other possible ways of generalizing the notion of permanent to the multidimensional case.

In Section 3, we consider the most important results of the theory of the permanents of two-dimensional matrices: the Bregman and Schrijver theorems on the bounds for the permanent of a $(0, 1)$-matrix, the König–Frobenius criterion for the positivity of the permanent, the van der Waerden conjecture and its proofs, and also the Birkhoff's theorem for doubly stochastic matrices.

In Section 4, we deal with upper bounds for the permanents of multidimensional $(0, 1)$-matrices in terms of the number of 1s in the planes and give an asymptotically attainable upper bound for the permanents of polystochastic matrices.

The remaining properties of polystochastic matrices are investigated in Section 5. Therein, we discuss how theorems on the permanents of doubly stochastic matrices are extended in passing to matrices of larger dimension, expose several constructions of polystochastic matrices, and find properties of polystochastic matrices of a special kind. Moreover, we give other ways of generalizing the notion of stochasticity to multidimensional matrices and their properties.

In Section 6, we continue the study of relationship between the number of transversals in Latin hypercubes and the permanents of multidimensional matrices. Here we give the exact values and bounds for the number of transversals in some Latin hypercubes, an asymptotically attainable upper bound for the number of transversals in Latin hypercubes and prove that every Latin hypercube of even order has even number of transversals.

In Section 7, we list several nontrivial sufficient conditions for the positivity of the permanent of a multidimensional matrix. Most of them are obtained as a consequence of rather difficult theorems on sufficient conditions for the existence of 1-factors of hypergraphs.

In Section 8, we highlight the most important questions and problems in the theory of multidimensional permanents.

1. DEFINITIONS, BASIC PROPERTIES, AND EXAMPLES

Let $n, d \in \mathbb{N}$ and let

$$I_n^d = \{(\alpha_1, \dots, \alpha_d) \mid \alpha_i \in \{1, \dots, n\}\}$$

be the index set. By a *d-dimensional matrix A of order n* we mean an array of numbers $(a_\alpha)_{\alpha \in I_n^d}$, $a_\alpha \in \mathbb{R}$.

For $k \in \{0, \dots, d\}$, a *k-dimensional plane* of a matrix A is a k-dimensional submatrix in A for which the values of $d - k$ coordinates are fixed and the remaining k coordinates range over all n values. A $(d-1)$-dimensional plane of a d-dimensional matrix is referred to as a *hyperplane*.

Refer as a *direction* Θ of a plane of a matrix to a $(0,1)$-vector of length d such that if, in this plane, the ith coordinate is variable then the corresponding coordinate in Θ is equal to 0, and if it is fixed then the ith coordinate in Θ is equal to 1.

We say that two multidimensional matrices are *equivalent* if they can be taken to each other by permutations of coordinates (transpositions) and permutations of hyperplanes inside one direction.

Given a d-dimensional matrix A of order n and an index $\alpha \in I_n^d$, the *minor of an element* a_α is the d-dimensional matrix $(A|\alpha)$ of order $n-1$ obtained from A by removing all entries lying in one hyperplane with a_α. In other words, $(A|\alpha)$ is constructed by removing from A the entries a_β such that $\alpha_i = \beta_i$ for some $i \in \{1, \dots, d\}$.

Let A be a d-dimensional matrix of order n. Then $D(A)$ stands for the set of all *diagonals* of A:

$$D(A) = \big\{(\alpha^1, \dots, \alpha^n) \mid \alpha^i \in I_n^d,\ \forall i \neq j,\ \rho(\alpha^i, \alpha^j) = d\big\},$$

where ρ is the Hamming distance (the number of positions at which two vectors differ). In other words, a diagonal is a set of n indices in the matrix such that each hyperplane contains exactly one index. The *permanent* of a matrix A is

$$\mathrm{per} A = \sum_{p \in D(A)} \prod_{\alpha \in p} a_\alpha.$$

Let us define various classes of multidimensional matrices:

For brevity, refer to the matrices whose all entries are 0s and 1s as $(0,1)$-*matrices*. A *diagonal* matrix is a $(0,1)$-matrix for which the indices of the unity entries form a diagonal.

A matrix A is called *nonnegative* if $a_\alpha \geq 0$ for $\alpha \in I_n^d$. Almost all matrices under consideration in this article are nonnegative. Refer as the *support* of a matrix to the set of the indices of its nonzero entries.

A nonnegative two-dimensional matrix such that the sum of the entries at each row and each column is equal to one is called *doubly stochastic*. The set of all doubly stochastic matrices of order n is denoted by Ω_n. The permanents of doubly stochastic matrices are the most studied objects in the theory of two-dimensional permanents.

There are several ways of generalizing the notion of doubly stochasticity to multidimensional matrices. The following definition will be regarded as the basic one: Refer to a nonnegative multidimensional matrix as *polystochastic* if the sum of the entries in each of its one-dimensional planes is equal to one.

Note that each two-dimensional plane in a polystochastic matrix is a doubly stochastic matrix. Denote the set of all d-dimensional polystochastic matrices of order n by Ω_n^d.

Refer to a multidimensional $(0,1)$-matrix as *transitive* if, by permutations of coordinates and permutations of hyperplanes inside one direction preserving the structure of the support of the matrix, every unit entry can be taken to every other unit entry. If a matrix is transitive then all its hyperplanes have the same number of 1s.

1.1. The Basic Properties of the Permanents of Multidimensional Matrices

Property 1. *The permanents of equivalent matrices coincide.*

Property 2. *The permanent is a linear function with respect to the hyperplanes of the matrix: under multiplying any hyperplane of a matrix by a number c, the permanent is also multiplied by c; the sum of the permanents of matrices differing by only one hyperplane is equal to the permanent of the matrix for which these hyperplanes are summed up.*

Property 3. *The following analog of the Laplace decomposition holds for multidimensional permanents: If* Γ *is a hyperplane in a d-dimensional matrix A of order* n *then*

$$\operatorname{per} A = \sum_{a_\alpha \in \Gamma} a_\alpha \operatorname{per}(A|\alpha).$$

Property 4. *Let A and B be nonnegative d-dimensional matrices of order* n *such that the entry* b_α *is at most* a_α *for every* $\alpha \in I_n^d$. *Then* $\operatorname{per} B \le \operatorname{per} A$.

Property 5. *The permanent of every transitive* $(0,1)$-*matrix divides by the number of nonzero entries in the hyperplanes.*

Proof. Since every unity entry of a transitive matrix can be taken to every other unity entry by transformations not changing the permanent of the matrix, the permanents of the minors of all unity entries are identical. By Property 3, the permanent of the matrix divides by the cardinality of the support of a hyperplane. Property 5 is proved. □

Property 6. *For the positivity of the permanent of a nonnegative d-dimensional matrix of order* n, *it is necessary that*

$$\sum_{i=1}^{d} x_i \le (d-1)n$$

for its every zero submatrix of size $x_1 \times \cdots \times x_d$. *Equality is attained at a diagonal matrix.*

Proof. Note that if the permanent of a matrix is positive then there exists a set of n nonzero entries piercing through all n hyperplanes of one direction. Consider a zero submatrix of size $x_1 \times \cdots \times x_d$. Then, for each $i = 1, \dots, d$, the number of hyperplanes of the ith direction disjoint from this submatrix is equal to $n - x_i$.

If there is a zero submatrix such that

$$\sum_{i=1}^{d} (n - x_i) < n$$

then all nonzero entries of the matrix can be covered by less than n hyperplanes, which contradicts the positivity of the permanent. Property 6 is proved. □

Property 7. *If a d-dimensional* $(0,1)$-*matrix A of order* n *contains exactly t zeros then*

$$\operatorname{per} A \ge (n^{d-1} - t)(n-1)!^{d-1}.$$

Proof. This property is a consequence of the fact that, through each entry in a d-dimensional matrix of order n, there pass exactly $(n-1)!^{d-1}$ diagonals. Property 7 is proved. □

Property 8. *The permanent of a multidimensional matrix can be expressed as the sum of the permanents of matrices of lesser dimensions as follows:*

Let A be some d-dimensional matrix of order n and let $1 \le k \le d-2$. Fix a direction of k-dimensional planes and enumerate all planes of this direction by $(d-k)$-dimensional indices. Define Σ as the set of mappings $\sigma : \{1, \dots, n\} \to I_n^{d-k}$ such that $(\sigma(1), \dots, \sigma(n))$ is a diagonal in a $(d-k)$-dimensional matrix. Then

$$\mathrm{per} A = \sum_{\sigma \in \Sigma} \mathrm{per} A_\sigma,$$

where A_σ are $(k+1)$-dimensional matrices of order n whose ith hyperplane is the $\sigma(i)$th k-dimensional plane of A.

Proof. Without loss of generality assume that the k-dimensional planes of A are defined by fixing the values of the first $d-k$ coordinates. By definition,

$$\mathrm{per} A = \sum_{p \in D(A)} \prod_{\alpha \in p} a_\alpha.$$

Partition the set of the diagonals $D(A)$ of A into the sets

$$D_\sigma(A) = \{p \in D(A) \mid p = ((\sigma(1), *, \dots, *), \dots, (\sigma(n), *, \dots, *))\},$$

where $*$ can stand for arbitrary elements of $\{1, \dots, n\}$. Regroup the summands in the sum defining the permanent:

$$\mathrm{per} A = \sum_{\sigma \in \Sigma} \sum_{p \in D_\sigma(A)} \prod_{\alpha \in p} a_\alpha.$$

It remains to observe that, for fixed σ, the quantity

$$\sum_{p \in D_\sigma(A)} \prod_{\alpha \in p} a_\alpha$$

is equal to the permanent of A_σ. Property 8 is proved. □

Property 9. *Let A be a nonnegative d-dimensional matrix of order n and let r_β be the sum of the entries of A in one-dimensional planes of some direction, $\beta \in I_n^{d-1}$. Consider the $(d-1)$-dimensional matrix B of order n with entries r_β. Then*

$$\mathrm{per} A \le \mathrm{per} B.$$

Proof. Assume without loss of generality that the sums in the one-dimensional planes of A are known for which all coordinates but the last are fixed. By definition,

$$\mathrm{per} B = \sum_{p \in D(B)} \prod_{\beta \in p} r_\beta = \sum_{p \in D(B)} \prod_{\beta \in p} \left(\sum_{i=1}^{n} a_{\beta,i} \right).$$

After removing parentheses, we can rewrite this expression as

$$\mathrm{per} B = \sum_{p \in D(B)} \left(\prod_{i=1,\dots,n,\ \beta_i \in p} a_{\beta_i,i} + R(p) \right) = \mathrm{per} A + \sum_{p \in D(B)} R(p),$$

where the summand $R(p)$ comprises the summands of the form

$$\prod_{i=1,\dots,n,\ \beta_i \in p} a_{\beta_i,k_i}$$

in which the numbers k_i can be repeated.

Since A is nonnegative, $\sum_{p \in D(B)} R(p)$ is at least zero, and so $\mathrm{per} B \ge \mathrm{per} A$. Property 9 is proved. □

1.2. Some Examples

Example 1. Refer as the *uniform matrix* J_n^d to the d-dimensional matrix of order n whose each entry is equal to $1/n$. Obviously, J_n^d is a polystochastic matrix.

The number of diagonals in every d-dimensional matrix of order n is equal to $n!^{d-1}$. By Property 2, the permanent of the uniform matrix is n^n times less than the number of all diagonals. Hence,

$$\mathrm{per} J_n^d = \frac{n!^{d-1}}{n^n}.$$

Example 2. Let E_n^d be the d-dimensional matrix of order n whose every element is equal to one, and let D_n^d be the diagonal d-dimensional matrix of order n. Find the permanent of $E_n^d - D_n^d$ by the inclusion-exclusion principle: From the set of all diagonals of E_n^d, exclude the diagonals passing through one of the entries of the unit diagonal D_n^d, and then add the diagonals passing exactly through two entries of this diagonal, etc. Thus,

$$\mathrm{per}\big(E_n^d - D_n^d\big) = \sum_{i=0}^{n} (-1)^i C_n^i (n-i)!^{d-1}.$$

This formula was firstly obtained in [8].

Example 3. Find the permanent of all polystochastic matrices of order 2. Since the sum of the entries in every one-dimensional plane of a polystochastic matrix is equal to one, the entries of such matrices of order 2 can take only two values. Let $A_2^d(x)$ be the d-dimensional matrix of order 2 whose entry a_α is equal to x if an index α contains evenly many 1s, and a_α is equal to $1-x$ if α contains oddly many 1s, $0 \le x \le 1$.

Each diagonal of the matrix $A_2^d(x)$ is a set of two indices taken one into another by replacing every one by two and vice versa.

If the dimension d is even then the number of ones in the indices from a diagonal has the same parity, and hence each diagonal in $A_2^d(x)$ has two identical entries. By symmetry, one half of the diagonals of this matrix makes a contribution equal to x^2 to the permanent; and the second half makes a contribution equal to $(1-x)^2$. Therefore, for an even dimension, we have

$$\mathrm{per} A_2^d(x) = 2^{d-2}(x^2 + (1-x)^2).$$

The maximum of the permanent of the even-dimensional matrices $A_2^d(x)$ equals 2^{d-2} and is attained at the $(0,1)$-matrix while the minimum is attained at the uniform matrix and equals 2^{d-3}.

If the dimension d is odd then the numbers of 1s in the indices from a diagonal have different parities. Therefore, the diagonal of $A_2^d(x)$ contains both x and $1-x$. Thus, for an odd dimension, we have

$$\mathrm{per} A_2^d(x) = 2^{d-1} x(1-x).$$

The maximum of the permanents of the odd-dimensional matrices $A_2^d(x)$ is attained at the uniform matrix and equals 2^{d-3}, while the minimum is attained at the $(0,1)$-matrix and is zero.

Example 4. Construct the set of $(0,1)$-matrices with zero permanent for which one half of the entries in every one-dimensional plane is equal to one. This example was earlier given in [13, 14].

Suppose that d is odd and $n \equiv 2 \mod 4$. Consider the d-dimensional $(0,1)$-matrix A of order n such that its entry a_α is equal to one if and only if $\sum_{i=1}^d \alpha_i$ is even. Note that the number of zeros in every one-dimensional plane of A is equal to the number of 1s.

Suppose that the permanent of A is greater than zero and $(\alpha^1, \dots, \alpha^n)$ is a unit diagonal in A. Consider the sum

$$S = \sum_{j=1}^{n} \sum_{i=1}^{d} \alpha_i^j.$$

On the one hand, S is even because, by the definition of A, the quantity $\sum_{i=1}^{d} \alpha_i^j$ is even for every $j = 1, \dots, n$. On the other hand, since all α_i^j are different for fixed i, we have

$$S = \sum_{i=1}^{d} \sum_{j=1}^{n} \alpha_i^j = \sum_{i=1}^{d} \frac{n(n+1)}{2} = \frac{dn(n+1)}{2}$$

and S is odd; a contradiction.

2. APPLICATIONS OF THE MULTIDIMENSIONAL PERMANENT IN COMBINATORICS

In this section, we describe the constructions of multidimensional matrices whose permanents are connected with various combinatorial objects. The relationship between the permanents of multidimensional matrices and transversals in Latin hypercubes is considered in more detail in Section 6. Section 7 contains several assertions on permanents which are obtained as corollaries to the theorems about hypergraphs.

2.1. 1*-Factors of* d*-Uniform Hypergraphs*

Let us study the relationship between the multidimensional permanent and 1-factors of d-uniform hypergraphs and, using it, express the number of some combinatorial objects. Before that, recall some necessary definitions concerning hypergraphs.

As is known, a pair $H = (X, W)$ is called a *hypergraph* with vertex set X and hyperedge set W, where each hyperedge $w \in W$ is a subset of vertices in X. A hypergraph H is called d*-uniform* if each of its hyperedges consists exactly of d vertices.

The *degree* of a vertex $x \in X$ in a hypergraph H is the number of the hyperedges $w \in W$ containing x. A hypergraph is called *regular* if all its vertices have the same degree.

To every hypergraph $H(X, W)$, we can uniquely assign the bipartite graph $B(X, W; E)$ whose one part is the vertex set X and the second part is the set of hyperedges W. Two vertices x and w in B are joined by an edge whenever the vertex x belongs to the hyperedge w in H. We call this graph $B(H) = B(X, W; E)$ the *bipartite representation* of H.

For a hypergraph $H(X, W)$, the *dual hypergraph* $H^*(W, H)$ is the hypergraph whose vertex set coincides with the hyperedge set of H, the hyperedge set of H^* is the vertex set of H, and a vertex w belongs to a hyperedge x in H^* if the hyperedge w contains the vertex x in H.

The notions of regularity and uniformity are dual in the following sense: If a hypergraph H is d-regular then the dual hypergraph H^* is d-uniform, and vice versa, if H is a d-uniform hypergraph then H^* is d-regular. It is easy that if a hypergraph H is d-uniform then the degree of every vertex w in the part W in the bipartite representation of B is equal to d, and if H is d-regular then the degree of every vertex x in the part X in B is equal to d.

Refer as an *orientation* of a hyperedge $w \in W$ in a hypergraph H to an ordering of the vertices of the hyperedge. A hypergraph H is *oriented* if an orientation is chosen for each of its hyperedges. An orientation of H is called *proper* if no vertex is in the same position in different orientations of the hyperedges of H.

By a k*-factor* of a hypergraph H we mean a k-regular subhypergraph covering all vertices of H. Denote by $\varphi(H)$ the number of 1-factors of H. Note that, for the existence of at least one 1-factor of a d-uniform hypergraph H on n vertices, it is necessary that n is divisible by d. 1-Factors of hypergraphs are a natural generalization of the notion of a perfect matching in graphs.

A hypergraph H is called *simple* if H does not contain multiple hyperedges. The *adjacency matrix* $A(H)$ of a simple d-uniform hypergraph H on n vertices is the d-dimensional $(0, 1)$-matrix of order n such that its entry a_α is one if and only if the vertices with labels from α constitute a hyperedge of H.

This definition of adjacency matrix of a hypergraph generalizes that of the adjacency matrix of a graph. It is not hard to see that the permanent of the adjacency matrix of a graph is equal to the number of spanning properly oriented 2-regular graphs composed of the edges of the original graph. In turn, the permanent of the adjacency matrix of a d-uniform hypergraph is equal to the number of spanning properly oriented d-regular hypergraphs constructed with the use of its hyperedges (each hyperedge can be used several times).

It is known that the number of ones in a row or in a column of the adjacency matrix of a graph is equal to the degree of the corresponding vertex. For the adjacency matrices of hypergraphs, the analogous property holds with respect to hyperplanes: The number of ones in the ith hyperplane of the matrix $A(H)$ equals $(d-1)!\deg(x_i)$, where $\deg(x_i)$ is the degree of the vertex x_i.

The dual objects to 1-factors of d-uniform hypergraphs are the transversals in d-regular hypergraphs. A *transversal* in a d-regular hypergraph H is a set of vertices piercing once through every hyperedge. To transversals in H, there uniquely correspond the 1-factors of the dual hypergraph H^*.

A vertex $x \in X$ in a hypergraph H is called *multiple* if there exists a vertex $y \in X$ different from x such that the sets of hyperedges incident to x and y coincide.

Let H is a d-regular hypergraph without multiple vertices with n hyperedges. Then for H we can construct the *hyperedge adjacency matrix* $A^*(H)$, which is the d-dimensional $(0,1)$-matrix of order n such that its entry a_α is equal to 1 if the intersection of the hyperedges with labels from α contains exactly one vertex of H. Note that the hyperedge adjacency matrix of a hypergraph H is the vertex adjacency matrix for the dual hypergraph H^*, and vice versa:

$$A^*(H) = A(H^*), \quad A(H) = A^*(H^*).$$

Thus, all relations between the number of 1-factors of a hypergraph and the permanent of its adjacency matrix listed below can be reformulated in terms of the number of its transversals and the permanent of the adjacency matrix of the dual hypergraph.

Let us pass to estimating the number of 1-factors of hypergraphs with the use of the multidimensional permanent.

Firstly, the number of 1-factors of every d-uniform hypergraph H does not exceed the permanent of its adjacency matrix. Indeed, let a set of hyperedges $e_1, \dots e_{n/d}$ be a 1-factor of H. For each hyperedge, fix some orientation α^i, $i = 1, \dots, n/d$. Then, using cyclic shifts of vertices in each of the orientations, construct tuples $\alpha^1, \dots, \alpha^n$. By the definition of the adjacency matrix $A(H)$, we conclude that its entry a_{α^i} is equal to 1 for each $i = 1, \dots, n$. The Hamming distance between different tuples α^i and α^j is d by construction. Consequently, the tuples $\alpha^1, \dots, \alpha^n$ form a unit diagonal in the adjacency matrix $A(H)$, and hence $\varphi(H) \le \operatorname{per} A(H)$.

This implies that the permanent of the adjacency matrices of d-uniform hypergraphs containing a 1-factor is always positive. In Section 7, we list several classes of matrices the positivity of whose permanent stems from the existence of a 1-factor of the corresponding hypergraph.

In [15], Alon and Friedland prove for graphs a substantially stronger inequality between the number of 1-factors and the permanent of the adjacency matrix, which is attained at a bipartite graph. This result can also be found in [16]:

Theorem 1. *Let G be a simple graph on n vertices. Then*

$$\varphi(G) \le (\operatorname{per} A(G))^{1/2}.$$

An analogous inequality for hypergraphs was proved in [17]:

Theorem 2. *Let H be a simple d-uniform hypergraph on n vertices, where d is a divisor of n, and let $\mu(n,d)$ be the function such that*

$$\mu(n,d) = \frac{d!^{2n}}{d^{dn} d!^{n/d}}$$

for all $d \ge 4$. Then the number of 1-factors of H satisfies the estimate

$$\varphi(H) \le \left(\frac{\operatorname{per} A(H)}{\mu(n,d)}\right)^{1/d}.$$

Since the number of ones in the hyperplanes of the adjacency matrix is connected with the degrees of the hypergraph; therefore, estimating its permanent in terms of the sums of the entries in the hyperplanes, we can bound the number of 1-factors via the vertex degrees. For example, in [15], for estimating the permanent of the adjacency matrix of a graph, the Bregman theorem [18] (Theorem 7) is used and the following assertion is proved:

Theorem 3. *Suppose that $G = (V, E)$ is a simple graph and d_i is the degree of a vertex $v_i \in V$. Then the number of* 1*-factors of G satisfies the inequality*

$$\varphi(G) \leq \prod_{i=1}^{n} d_i!^{1/2d_i}.$$

This result was earlier obtain by Kahn and Lovas but their proof was not published, and it is proved in [19] by the entropy method.

Applying the trivial upper bound of Section 4 for the permanent of a multidimensional matrix in Theorem 2, we obtain

Corollary 1 [17]. *Suppose that $H = (X, W)$ is a simple d-uniform graph and r_i are the degrees of vertices $x_i \in X$. Then the number of* 1*-factors of H satisfies the inequality*

$$\varphi(H) \leq \left(\frac{(d-1)!^n}{\mu(n,d)} \prod_{i=1}^{n} r_i \right)^{1/d}.$$

In conclusion, give a list of objects whose number is naturally expressed as the number of 1-factors of a hypergraph of a special kind, and hence the multidimensional permanents can be used for their estimation.

1. A *Steiner system with parameters t, k, and n* ($t < k < n$) is a set of k-element blocks in some n-element set X such that each t-element subset in X lies exactly in one block. Let $S(t, k, n)$ be the number of Steiner systems with parameters t, k, and n. For calculating this number, consider the D-uniform hypergraph $H_S(t, k, n)$ on N vertices, where $D = C_k^t$ and $N = C_n^t$. The vertices of $H_S(t, k, n)$ are the t-element subsets in the n-element set X, and the hyperedge consists of D vertices such that their union gives a k-element set. Then the number of 1-factors of $H_S(t, k, n)$ coincides with the number of Steiner systems:

$$\varphi(H_S(t, k, n)) = S(t, k, n).$$

The problem of the existence of Steiner systems with given parameters and the estimation of their number is the topic of many studies. For example, an upper bound for the number of Steiner triple systems was proved in [20].

2. Consider a graph $G(V, E)$ such that all its maximal cliques are of cardinality d. The number of maximal cliques will be called a *perfect clique matching* if the union of these cliques covers all vertices of the graph one time. Let $\mathrm{PCM}(G)$ denote the number of perfect click matchings in G. From the graph G, construct the d-uniform hypergraph $H_{PCM}(G)$ such that the vertices of this hypergraph are the vertices of G, and the hyperedges of $H_{PCM}(G)$ correspond to the maximal cliques of G. It is easy that the number of 1-factors of $H_{PCM}(G)$ is equal to the number of perfect click matchings in G:

$$\varphi(H_{PCM}(G)) = \mathrm{PCM}(G).$$

3. Let X be a set of kd points in $\mathbb{R}^{d-1}$, $k \geq 1$. A point $p \in \mathbb{R}^{d-1}$ is called a *Birch point* for X if there exists a partition of X into k subsets of d points such that the convex hull of each of k subsets contains p. Refer to a partition of X with this property as a *Birch partition* with respect to p and denote by $BP(X, p)$ the number of Birch partitions.

Construct the d-uniform hypergraph $H_{BP}(X, p)$ for a set of points X and a point p in $\mathbb{R}^{d-1}$ so that the vertex set of the hypergraph is X and d points are united in a hyperedge if their convex hull contains p. Then the number of 1-factors of $H_{BP}(X, p)$ is equal to the number of Birch partitions of X with respect to p:

$$\varphi(H_{BP}(X, p)) = BP(X, p).$$

Some results on Birch partitions are exposed in [21].

2.2. 1-Factors of a d-Partite Hypergraph

In Section 2.1, we demonstrated that the number of 1-factors of any d-uniform hypergraph is estimated via the multidimensional permanent. Show that, for d-partite hypergraphs, there exists a multidimensional matrix such that its permanent is equal to the number of 1-factors.

A hypergraph $H = (X, W)$ is called *d-partite* if the set of its vertices can be partitioned into d subsets (parts) such that each hyperedge contains exactly one vertex from each subset. Obviously, every d-partite hypergraph is d-uniform. A d-partite hypergraph is called *balanced* if all its parts are of the same cardinality.

The d-partiteness of $H(X, W)$ is equivalent to the fact that the part W of its bipartite representation $B(X, W; E)$ splits into the semiperfect codes, where a *semiperfect code* in a bipartite graph is a subset of vertices in one of its parts such that the neighborhoods of different vertices in this subset are disjoint and the union of all neighborhoods coincides exactly with the second part of the graph.

Let H be a d-partite graph with parts of cardinality n. Refer as the *adjacency matrix* $\widetilde{A}(H)$ *of parts of* H to the d-dimensional matrix of order n such that its entry a_α is equal to the multiplicity of the hyperedge $(\alpha_1, \dots, \alpha_n)$ in H, where α_i is a vertex in the part i. Thus, there is a one-to-one correspondence between the nonnegative integer d-dimensional matrices and the balanced d-partite hypergraphs and, as it is not hard to see, the permanent of the adjacency matrix $\widetilde{A}(H)$ of parts is equal to the number of 1-factors of H. The correspondence between the number of 1-factors of a simple d-partite graph and the permanent of a multidimensional $(0, 1)$-matrix was discovered by Dow and Gibson in [8].

As was shown in Section 2.1, the transversals are the objects dual to 1-factors of a hypergraph, and so, for suitable hypergraphs, we can construct a multidimensional matrix whose permanent is equal to the number of transversals. Let us describe this construction in detail.

Let H be a δ-regular hypergraph partitioned into disjoint 1-factors. Since H is δ-regular, the degree of every vertex x in the part X in its bipartite representation B is equal to δ, and since it additionally splits into 1-factors, the part X of its bipartite representation splits into the semiperfect codes.

Suppose that every 1-factor of a hypergraph H contains the same number of hyperedges and denote their number by η. Refer as the 1-*factor adjacency matrix* to the δ-dimensional matrix $\widetilde{A}^*(H)$ of order η such that its entry a_α equals the number of vertices in the intersection of the hyperedges α_i of the ith 1-factor over all $i = 1, \dots, \delta$. Then the permanent of the 1-factor adjacency matrix $\widetilde{A}^*(H)$ is equal to the number of transversals in H. Moreover, to each δ-dimensional integer matrix with nonnegative entries, there corresponds a δ-regular hypergraph such that its permanent is equal to the number of transversals in the hypergraph; and vice versa.

There are many combinatorial objects whose number is equal to the quantity of 1-factors of a balanced d-partite hypergraph of a special kind, and hence multidimensional permanents can be useful for estimating this number. Most important among these objects are the tilings of the transitive graphs, MDS codes, Latin squares, and Latin hypercubes. Finish this subsection by describing the appropriate constructions of multidimensional matrices and hypergraphs.

1. The first construction was proposed in [22]. Let $G(V, E)$ be a regular graph. Denote by $\mathrm{Aut}(G)$ the automorphism group of G (the group of mappings of the vertices of the graph onto itself which preserve adjacency). A graph G is called *transitive* if the group $\mathrm{Aut}(G)$ acts transitively on its vertex set; i.e., for every $u, v \in V$ there exists an automorphism $a \in \mathrm{Aut}(G)$ such that $u = a(v)$.

Choose a set of vertices S of cardinality d in a regular transitive graph G and call it a *tile*. A set of automorphisms $\{a_i\}_{i=1}^n$ is called a *tiling* of G by S if $a_i(S) \cap a_j(S) = \varnothing$ for all $i \neq j$, and for each vertex $v \in V$ there is an automorphism a_i such that v belongs to $a_i(S)$. Denote by $T(G, S)$ the number of tilings of G by S.

Construct the d-uniform hypergraph $H_T(G, S)$ whose vertices are all vertices of G, and each hyperedge is the set of vertices $a(S)$ for some automorphism $a \in \mathrm{Aut}(G)$. If $H_T(G, S)$ is d-partite (i.e., if its vertices can be partitioned into d groups of cardinality n so that the intersection of all $a(S)$ with every of these groups is nonempty) then for it we can construct the adjacency matrix $A_T(G, S)$ of parts, which has dimension d and order n and whose permanent coincides with the number of tilings of the graph G by the tile S:

$$\mathrm{per} A_T(G, S) = T(G, S).$$

Most of the constructions below are particular cases of the given construction.

2. A *tiling* of a finite abelian group $\langle R, +\rangle$ is a pair of subsets S and T such that $S + T = R$ and $|S||T| = |R|$. Denote by s the cardinality of S and designate as $GT(R, S)$ the number of tilings of the group R by the tile S. Consider the hypergraph $H_{GT}(R, S)$ whose vertex set is the set of the elements of R and whose hyperedges are the sets of the form $x - S$ for some element $x \in R$. Note that the number of hyperedges in this hypergraph is equal to the number of its vertices and that it is both s-uniform and s-regular. If in R there exists at least one tiling (S, T) by S then the hypergraph $H_{GT}(R, S)$ is s-partite and the sets $s + T$, $s \in S$ can be taken as the parts. To every tiling of the group R by a tile S, one can assign both a 1-factor in $H_{GT}(R, S)$ and a transversal. Thus, the number of tilings of R is equal to the permanent of the adjacency matrix $A_{GT}(R, S)$ of parts of dimension s and order $|R|/s$ and to the permanent of the 1-factor adjacency matrix $A^*_{GT}(R, S)$ of the same dimension and order:

$$GT(R, S) = \mathrm{per} A_{GT}(R, S) = \mathrm{per} A^*_{GT}(R, S).$$

3. Let $\mathbb{E}^n$ be the Boolean hypercube of dimension n. A *perfect code* C in $\mathbb{E}^n$ is a subset such that each ball of unit radius in $\mathbb{E}^n$ contains exactly one element of C.

It is not hard to see that the cardinality of every perfect code C equals $N = 2^n/(n + 1)$. Perfect codes exist only in Boolean hypercubes $\mathbb{E}^n$ with $n = 2^m - 1$; moreover, every of these cubes can be partitioned into $D = n + 1$ disjoint perfect codes. Let $PC(n)$ denote the number of perfect codes in $\mathbb{E}^n$.

Observe that the minimum distance graph in the hypercube $\mathbb{E}^n$ is a transitive graph. As the tile in it consider the ball of unit radius and construct a D-uniform hypergraph $H_{PC}(n)$ the number of 1-factors of which is equal to the number of tilings of the hypercube by balls of unit radius. Since $\mathbb{E}^n$ splits into perfect codes, $H_{PC}(n)$ is D-partite; and if we take only the shifts by nonzero vectors as the automorphisms of $\mathbb{E}^n$ then $H_{PC}(n)$ is simple. The adjacency matrix $H_{PC}(n)$ of parts is a D-dimensional (0,1)-matrix of order $N = 2^n/(n + 1)$, and the number of all perfect codes in the Boolean hypercube $\mathbb{E}^n$ coincides with its permanent:

$$\mathrm{per} A_{PC}(n) = PC(n).$$

4. Let $\mathbb{Z}_q^n$ denote the n-dimensional hypercube of order q. A subset in $\mathbb{Z}_q^n$ of cardinality q^{n-d+1} in which the minimal Hamming distance between the elements is d is called an MDS *code with distance* d. In other words, each $(d - 1)$-dimensional plane in $\mathbb{Z}_q^n$ contains only one element of an MDS code. Let $MDS(n, q, d)$ stand for the number of MDS codes with distance d in $\mathbb{Z}_q^n$.

Construct the hypergraph $H_{MDS}(n, q, d)$ whose vertices are all $(d - 1)$-dimensional planes in $\mathbb{Z}_q^n$, and each hyperedge is a collection of $D = C_n^{d-1}$ $(d - 1)$-dimensional planes such that their intersection contains exactly one element of $\mathbb{Z}_q^n$. The so-obtained hypergraph is simple and D-partite, where as the parts we can take all $(d - 1)$-dimensional planes of one direction.

The adjacency matric $A_{MDS}(n, q, d)$ of parts of this hypergraph is a D-dimensional $(0, 1)$-matrix of order $N = q^{n-d+1}$, and its permanent is connected with the number of MDS codes with distance d in $\mathbb{Z}_q^n$:

$$\mathrm{per} A_{MDS}(n, q, d) = MDS(n, q, d).$$

5. Consider the n-dimensional hypercube $\mathbb{Z}_q^n$ of order q. Let $PMDS(n, q, d)$ denote the number of partitions of $\mathbb{Z}_q^n$ into MDS codes with distance d. Construct the simple hypergraph $H_{PMDS}(n, q, d)$ whose vertices are the vertices of $\mathbb{Z}_q^n$ and whose hyperedges correspond to the MDS codes with distance d in $\mathbb{Z}_q^n$. If such MDS codes exist then the hypergraph $H_{PMDS}(n, q, d)$ is nonempty and q^{n-d+1}-partite, where the sets of vertices in $(d - 1)$-dimensional planes of one direction can be taken as the parts.

Let $A_{PMDS}(n, q, d)$ be the adjacency matrix of parts of this hypergraph. The dimension D of $A_{PMDS}(n, q, d)$ is q^{n-d+1}, and its order N equals q^{d-1}. The permanent of the $(0, 1)$-matrix $A_{PMDS}(n, q, d)$ is equal to the number of partitions of $\mathbb{Z}_q^n$ into MDS codes with distance d:

$$\mathrm{per} A_{PMDS}(n, q, d) = PMDS(n, q, d).$$

This construction can in particular be used for calculating the number of Latin hypercubes. A *Latin hypercube* of dimension n and order q is a filling of $\mathbb{Z}_q^n$ by the numbers $\{1, \dots, q\}$ such that, in every one-dimensional plane, all numbers are distinct. Latin hypercubes of dimension 2 are usually called *Latin squares*.

Each n-dimensional Latin hypercube of order q can be regarded as an ordered partition of $\mathbb{Z}_q^n$ into MDS codes with distance 2. If we denote by $LC(n,q)$ the number of n-dimensional Latin hypercubes of order q then

$$LC(n,q) = q!PMDS(n,q,2) = q!\,\mathrm{per} A_{PMDS}(n,q,2).$$

In particular, the number of Latin squares of order q is expressed through the permanent of the q-dimensional $(0,1)$-matrix $A_{PMDS}(2,q,2)$ of order q whose entry a_α is equal to one if the set of indices $\{(i,\alpha_i)\}_{i=1}^q$ is a diagonal in the square $\mathbb{Z}_q^2$ of order q:

$$LC(2,q) = q!PMDS(2,q,2) = q!\,\mathrm{per} A_{PMDS}(2,q,2).$$

For estimating the number of Latin rectangles of size $k \times q$, we can construct a k-dimensional matrix of order q with a structure similar to $A_{PMDS}(2,q,2)$. To unit elements of this matrix, there correspond collections of k elements of the rectangle $k \times q$ situated in different rows and columns.

An asymptotic upper bound for the number of Latin hypercubes is proved in [10].

6. Using the multidimensional permanent, we can express not only Steiner systems but also the number of H- and A-designs, which are their generalizations as was shown by Potapov in [23].

Let $n \geq w \geq t$. Refer as a *design of type* $H(n,q,w,t)$ to a collection of $(n-w)$-dimensional planes of $\mathbb{Z}_q^n$ such that each $(n-t)$-dimensional plane in $\mathbb{Z}_q^n$ contains exactly one plane of this collection. Note that a design $H(n,q,n,t)$ is an MDS code with distance $d = n - t + 1$. A *design of type* $A(n,q,w,t)$ is a collection of $(n-t)$-planes of $\mathbb{Z}_q^n$ that covers once all its $(n-w)$-dimensional planes.

Consider a simple q^{w-t}-uniform hypergraph $H_{AHD}(n,q,w,t)$ whose vertices are the $(n-w)$-planes of $\mathbb{Z}_q^n$ and whose hyperedges consist of q^{w-t} vertices such that the union of the corresponding planes in $\mathbb{Z}_q^n$ gives an $(n-t)$-dimensional plane. Then the number of 1-factors of $H_{AHD}(n,q,w,t)$ is equal to the number of A-designs $A(n,q,w,t)$, and the number of transversals is equal to the number of H-designs $H(n,q,w,t)$. If the hypercube $\mathbb{Z}_q^n$ splits into disjoint H-designs then the number of A-designs in this hypercube is equal to the permanent of the adjacency matrix of parts of $H_{AHD}(n,q,w,t)$; and vice versa, if $\mathbb{Z}_q^n$ splits into disjoint A-designs then the number of its H-designs is equal to the permanent of the 1-factor adjacency matrix of $H_{AHD}(n,q,w,t)$.

If $\mathbb{Z}_q^n$ does not split into A- or H-designs then their number can be estimated by the methods exposed in Section 2.1.

7. Let X be a set of nd colored points in $\mathbb{R}^{d-1}$, where the number of points of one color is exactly n and p is a point in $\mathbb{R}^{d-1}$. A *colored Birch partition* of X with respect to p is its partition into n disjoint sets such that each subset consists of d points of different colors and contains p in its convex hull. Denote by $CBP(X,p)$ the number of colored Birch partitions of a set X with respect to a point p.

Given a fixed set of colored points X and a point p, construct the simple d-uniform hypergraph $H_{CBP}(X,p)$ whose vertex set is X and a collection of d vertices of different colors constitutes a hyperedge if their convex hull contains p. Obviously, this hypergraph is d-partite, where the parts are the sets of points of one color.

The adjacency matrix $A_{CBP}(X,p)$ of parts of such a hypergraph has dimension d and order n, and its permanent is equal to the number of colored Birch partitions of the set X with respect to the point p; i.e.,

$$\mathrm{per} A_{CBP}(X,p) = CBP(X,p).$$

Some estimates of the number of colored Birch partitions can be found in [24].

2.3. Other Applications of Multidimensional Permanents

Not all combinatorial objects whose number can be expressed through the permanent of a multi-dimiensional matrix can be conveniently described in terms of 1-factors of hypergraphs. In this section, we consider several examples of these objects.

Recall that a d-dimensional *Latin hypercube* Q of order n is a d-dimensional matrix of order n containing exactly n different elements in which all entries of every one-dimensional plane are distinct. A two-dimensional Latin hypercube is often called a *Latin square*, and a three-dimensional Latin hypercube is called a *Latin cube*. A *transversal* in a Latin cube Q is a diagonal not containing identical elements. The following way of relating the multidimensional permanent and the number of transversals in a Latin hypercube was described in 1968 by Jurkat and Ryser [25], and its application will be considered in more detail in Section 6.

To each d-dimensional Latin hypercube Q, assign a $(d+1)$-dimensional $(0,1)$-matrix A of the same order by the following rule: An element of the hypercube q_α is equal to i if and only if the entry $a_{\alpha,i}$ of the matrix is equal to one. The permanent of such a matrix A is equal to the number of transversals in Q.

One more object whose enumeration is possible with the use of the multidimensional permanent is a 1-factorization of a hypergraph.

Refer as a 1-*factorization* of a hypergraph H to a partition of all its hyperedges into disjoint 1-factors and refer to a hypergraph in which there exists at least one 1-factorization as 1-*factorable*. Obviously, every 1-factorable graph is regular. Denote by $\Phi(H)$ the number of 1-factorizations of a hypergraph H.

There exist several ways of calculating the number of 1-factorizations of a regular hypergraph with the use of multidimensional permanents. Firstly, it can be obtained by consecutively removing 1-factors from a hypergraph H and estimating the number of available 1-factors at each step. In this way, one proves

Theorem 4 [17]. *Let G_n^d be a complete d-uniform graph on n vertices, i.e., a hypergraph whose set of hyperdges consists of all d-elements subsets of vertices. Then the number of its 1-factorizations satisfies the inequality*

$$\Phi(G_n^d) \leq \left((1+o(1))\frac{n^{d-1}}{\mu(n,d)^{1/n}e^d}\right)^{\frac{n^d}{d!}} \qquad as \quad n \to \infty$$

with the function $\mu(n,d)$ of Theorem 2.

Note that the existence of a 1-factorization for a complete d-uniform hypergraph on n vertices, where d divides n, was proved by Baranyai [26].

The second way to calculate the number of 1-factorization of a k-regular hypergraph H on n vertices consists in constructing a multidimensional matrix whose permanent coincides with this number.

For each vertex x_i of H, enumerate from 1 to k the hyperedges incident to this vertex. Construct an n-dimensional $(0,1)$-matrix $A_\Phi(H)$ of order k such that its entry a_α is equal to one if the union over all $i = 1,\dots,n$ of all hyperedges α_i incident to the ith vertex constitutes the 1-factor of H. Observe that the multiplicity of each hyperedge in a 1-factor is equal to k. The permanent of the constructed matrix coincides with the number of 1-factorizations of H:

$$\mathrm{per} A_\Phi(H) = \Phi(H).$$

1-Factorizations of a hypergraph, as well as 1-factors, can be applied to estimate other objects. For example, the number of symmetric Latin squares is expressed through the number of 1-factorizations.

A Latin square is called *symmetric* if it does not change under transposition. Let $SLS(n)$ denote the number of symmetric Latin squares of order n.

Consider a complete graph with loops K'_n on n vertices; assume in addition that each loop covers its vertex once. Then the adjacency matrix of K'_n is the unit matrix. Every partition of K'_n into 1-factors corresponds to a partition of its adjacency matrix into symmetric diagonals, which implies that the number of symmetric Latin squares is equal to the number of ordered 1-factorizations of K'_n:

$$SLS(n) = n!\Phi(K'_n) = n!\,\mathrm{per} A_\Phi(K'_n).$$

2.4. Other Multidimensional Generalizations of the Permanent

There exist several generalizations of the two-dimensional permanent to the multidimensional case. The main difference between them consists in different definitions of a diagonal.

In this article, unless otherwise specified, we consider the permanent in which by diagonals of a d-dimensional matrix we mean MDS codes with distance d. However, as diagonals we can also use MDS codes with other distances.

Let A be a d-dimensional matrix of order n. Define an r-*diagonal* of A as the set of indices of an MDS code with distance r and denote by $D_r(A)$ the set of all r-diagonals of A. The r-*permanent* of A is

$$\mathrm{per}_r A = \sum_{p \in D_r(A)} \prod_{\alpha \in p} a_\alpha.$$

For $r = d$, we obtain the definition of the permanent taken as the basic one in this article. As is known, only MDS codes with distances 2 and d exist in any d-dimensional matrix. Therefore, the case of $r = 2$ deserves special attention as well as the case of $r = d$. Moreover, the 2-permanent can find applications in enumeration problems as well. For example, the number of $(d-1)$-dimensional Latin hypercubes of order n is equal to the 2-permanent of the d-dimensional unit matrix E_n^d of order n. Estimating the 2-permanent of this matrix, Linial and Luria obtained

Theorem 5 [10]. *Let $LC(d,n)$ be the number of d-dimensional Latin hypercubes of order n. Then*

$$LC(d,n) \le \left((1+o(1))\frac{n}{e^d}\right)^{n^d} \quad \text{as} \quad n \to \infty.$$

In favor of the choice of the d-permanent as the main generalization of the two-dimensional permanent is the fact that the r-permanent of every d-dimensional matrix A of order n is equal to the d-permanent of a suitable multidimensional matrix. To show this, it suffices to consider the matrix $B_r(A)$ of dimension $D = C_d^{r-1}$ and order $N = n^{d-r+1}$ constructed by analogy to the matrix $A_{MDS}(d,n,r)$ expressing the number of MDS cods with distance r in the d-dimensional hypercube of order n, in which, instead of the unit entries of the matrix $A_{MDS}(d,n,r)$, we should take the entries of A lying in the intersection of the corresponding $(r-1)$-dimensional planes.

Such a construction was first described by Dow and Gibson [8], and, using it, they obtained the following bound for the 2-permanent:

Theorem 6 [7]. *Suppose that A is a three-dimensional $(0,1)$-matrix of order n and $r_{i,j}$ is the number of* 1s *in an (i,j)-dimensional plane of some direction. Then*

$$\mathrm{per}_2 A \le \prod_{i,j=1}^{n} r_{i,j}!^{1/r_{i,j}}.$$

In contrast to Bregman theorem (Theorem 7), which is formulated in Section 3, the bound for the 2-permanent in Theorem 6 is unattainable.

3. THE CLASSICAL THEOREMS ON TWO-DIMENSIONAL PERMANENTS

In this section, we give the most important assertions about the permanents of two-dimensional matrices. Unfortunately, many of these theorems do not extend to multidimensional matrices, and it is very hard to check the validity of other theorems for the multidimensional case.

Many lower and upper bounds were obtained for the permanents of two-dimensional matrices. The most famous upper bound is the following theorem, which was conjectured by Minc in 1963 in [27] and proved at different times and in different ways by Bregman [18], Schrijver [28], and Radhakrishnan [29].

Theorem 7 (Bregman). *Let A be a two-dimensional $(0,1)$-matrix of order n and let r_i be the number of* 1*s in the ith row of A. Then*

$$\operatorname{per} A \leq \prod_{i=1}^{n} r_i!^{1/r_i}.$$

Equality is attained at block-diagonal matrices.

Many other upper bounds for the two-dimensional permanent can be found in [2] or [30].

The positivity of the permanent of a two-dimensional nonnegative matrix is easily checked by means of the following criterion which can be proved by induction on the order of the matrix or obtained as a consequence of Hall's marriage theorem.

Theorem 8 (Frobenius, König). *Let A be a nonnegative two-dimensional matrix of order n. The permanent of A is zero if and only if A contains a zero $(s \times t)$-submatrix such that $s + t = n + 1$.*

One of the easy consequences of Theorem 8 is the fact that the permanent of a nonnegative matrix for which the sums of the entries in any row and column coincide is always positive. The best lower bound for the permanent of such matrices with integer entries is given by

Theorem 9 [31]. *Suppose that A is a nonnegative two-dimensional matrix of order n with integer entries and let the sum of entries in each row and each column of A be equal to k. Then*

$$\operatorname{per} A \geq \left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n.$$

Another important example of matrices with fixed sums of entries over rows and columns is doubly stochastic matrices. Recall that a nonnegative two-dimensional matrix is called *doubly stochastic* if the sum of the entries in each row and each column is equal to one. The set of doubly stochastic matrices forms a polytope in the space of all matrices, which is often called the *Birkhoff polytope*.

As was observed earlier, Theorem 8 implies

Theorem 10. *The permanent of every doubly stochastic matrix is positive.*

In fact, we have a somewhat stronger assertion:

Theorem 11 (Birkhoff). *Let A be a doubly stochastic matrix of order n. Then*

$$A = \sum_{i=1}^{k} \theta_i P_i,$$

where $P_1, \dots, P_k$ are diagonal matrices and $\theta_1, \dots, \theta_k$ are nonnegative numbers, $\sum_{i=1}^{k} \theta_i = 1$.

In other words, the angles of the Birkhoff polytope of doubly stochastic matrices are exactly the diagonal matrices.

Consider the problem of finding the minimum and maximum of the permanent among doubly stochastic matrices. It is not hard to understand that the maximum is attained at a diagonal $(0,1)$-matrix and is equal to one. The question of the minimum of the permanent for doubly stochastic matrices is much harder. The answer to this question is given by the following theorem, ascribed as a conjecture to van der Waerden and proved in 1980 by Egorychev [32] and Falikman [33]:

Theorem 12. *The minimum of the permanent in the class of doubly stochastic matrices of order n is equal to $n!/n^n$ and is attained only at the uniform matrix J_n whose each entry is $1/n$.*

Alternative proofs of this theorem were later found by Gyires [34, 35], Bapat [36], and Gurvits [37].

Describe the main ideas of the proofs of Egorychev and Falikman of the van der Waerden conjecture.

Firstly, both proofs use the fact that the permanent of a matrix is a multilinear function of the rows. For positive matrices $A = (A_1, \dots, A_n)$, where A_i is the ith row of A, the quadratic form

$$f(x, y) = \operatorname{per}(A_1, \dots, A_{n-2}, x, y)$$

of the variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is introduced and the validity of one of the following equivalent properties is proved:

Proposition 1. (i) *Let $s, t \in \mathbb{R}^n$ be real vectors, where $t \neq 0$. Suppose that $f(s,t) = 0$ and $f(s,s) > 0$. Then $f(t,t) < 0$.* (ii) *Let $s, t \in \mathbb{R}^n$ be real vectors, where all entries of s are nonnegative. Then $f^2(s,t) \geq f(s,s)f(t,t)$.*

Item (i) of Proposition 1 was proved by Falikman directly from the definition of the quadratic form f and the properties of the permanent of a matrix. Item (ii) was obtained by Egorychev as a corollary to the Alexandrov–Fenchel inequality for mixed discriminants.

Then, in the articles by these authors, Proposition 1 finds different applications. Falikman introduced the auxiliary matrix function

$$F_\varepsilon(A) = \mathrm{per}A + \varepsilon / \prod_{i,j=1}^{n} a_{i,j},$$

on the set of doubly stochastic matrices, where $\varepsilon \geq 0$ is an arbitrary real number. It is not hard to check that the minimum of $F_\varepsilon(A)$ is attained not on the boundary of the Birkhoff polytope but inside it. Using Proposition 1, Falikman proves that $F_\varepsilon(A)$ can attain its minimum only at the uniform matrix J_n.

Egorychev uses Proposition 1 for proving that, in any doubly stochastic matrix minimizing the permanent (and not only in a positive matrix, which was already proved in [38]), the permanent of each minor of every entry is equal to the permanent of the whole matrix. Granted the results obtained earlier by Marcus and Minc for doubly stochastic matrices and exposed in [2], it is not hard to deduce that this property is sufficient for the validity of the van der Waerden conjecture.

Unfortunately, for the multidimensional case, where it is natural to consider the hyperplanes instead of the rows, the analog of Proposition 1 fails. In the three-dimensional case, this can be illustrated as follows:

$$s = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\mathrm{per}^2(s,t) = \mathrm{per}^2 \begin{pmatrix} 1 & 1 & \times & -1 & 1 \\ 1 & 1 & & 1 & -1 \end{pmatrix} = 0,$$

$$\mathrm{per}(s,s)\mathrm{per}(t,t) = \mathrm{per} \begin{pmatrix} 1 & 1 & \times & 1 & 1 \\ 1 & 1 & & 1 & 1 \end{pmatrix} \mathrm{per} \begin{pmatrix} -1 & 1 & \times & -1 & 1 \\ 1 & -1 & & 1 & -1 \end{pmatrix} = 4 \cdot 4 = 16.$$

It is easy to generalize this example to the case of arbitrary dimension by filling the two-dimensional planes in the hyperplanes by zero matrices and matrices with s and t in an appropriate way.

Consider the approach of Gurvits [37] that is based on a fundamentally different idea and, alongside the van der Waerden conjecture, also makes it possible to prove the Schrijver's theorem (Theorem 9). This proof is also exposed in [39].

To a doubly stochastic matrix A, the polynomial is introduced

$$p(x_1, \dots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j} x_j \right).$$

Note that the coefficient at $x_1 \cdots x_n$ in $p(x_1, \dots, x_n)$ is equivalent to the permanent of A:

$$\frac{\partial^n p}{\partial x_1 \dots \partial x_n}(x_1, \dots, x_n) = \mathrm{per}A.$$

Together with $p(x_1, \ldots, x_n)$, consider the polynomial

$$p_i(x_1, \ldots x_i) = \frac{\partial^{n-i} p}{\partial x_{i+1} \ldots \partial x_n}\bigg|_{x_{i+1}=\cdots=x_n=0}.$$

Then $p_n(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$ and $p_0 = \mathrm{per}A$.

Then, for an arbitrary polynomial $f(x_1, \ldots, x_n)$, the notion of *capacity* is introduced:

$$\mathrm{cap}(f) = \inf_{x \in \Omega} f(x_1, \ldots, x_n), \quad \text{where} \quad \Omega = \left\{(x_1, \ldots, x_n) \mid \prod_{i=1}^{n} x_i = 1,\ x_i \in \mathbb{R}_+\right\}.$$

Using the inequality between the arithmetic and geometric means, Gurvits finds that the capacity of $p(x_1, \ldots, x_n)$ is equal to 1 for every doubly stochastic matrix. Since p_0 coincides with per$A$, its capacity is equal to the permanent.

Further, call a polynomial f of n variables *stable* if f has no zeros in $\mathbb{C}_+^n$, where $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$ is the right half-plane of the complex plane. Note that for every doubly stochastic matrix A the polynomial $p(x_1, \ldots, x_n)$ is stable.

With all above-introduced notions taken into account, the van der Waerden conjecture ensues from

Proposition 2. *Suppose that a polynomial $f(x_1, \ldots, x_n)$ of degree n is stable and homogeneous. Put $f' = \frac{\partial p}{\partial x_n}\big|_{x_n=0}$. Then either $f' \equiv 0$ or f' is stable and*

$$\mathrm{cap}(f') \geq \mathrm{cap}(f) g(\deg_{x_n} f),$$

where

$$g(0) = 1, \qquad g(y) = \left(\frac{y-1}{y}\right)^{y-1},$$

and $\deg_{x_n} f$ is the maximal degree of x_n in f.

The main difficulty for applying this method to the lower estimation of the permanent of multidimensional matrices is the construction from a polystochastic matrix A of a polynomial p with sufficiently good properties.

More detailed information about the permanents of two-dimensional matrices and related problems can be found in the classical book [2], and a modern survey of the result and open questions is presented in [40].

4. UPPER BOUNDS FOR THE MULTIDIMENSIONAL PERMANENT

Using induction and Property 9 of multidimensional permanents, it is easy to obtain the tirvial upper bound:

Theorem 13. *Let A be a d-dimensional nonnegative matrix of order n. Suppose that the sum of entries in the ith hyperplane of A is equal to r_i. Then*

$$\mathrm{per}A \leq \prod_{i=1}^{n} r_i.$$

In this section, we give several substantially subtler upper bounds on the permanents of polystochastic and $(0, 1)$-matrices.

4.1. Upper Bounds for the Permanents of $(0,1)$*-Matrices*

Slightly weakening most bounds for $(0,1)$-matrices, some estimates can be obtained for the permanent of an arbitrary nonnegative matrix whose all entries are at most 1.

Recall that, for the two-dimensional $(0,1)$-matrices, the Bregman theorem gives an upper bound for the permanent which is attained at all block-diagonal matrices and close to the lower bound for matrices with the same number of 1s in all rows and columns. It would be good to obtain an analogously sharp bound via the number of 1s for multidimensional matrices; namely, an exact bound or a bound close to the true value of the permanent in sufficiently many cases.

One of the first attempts to generalize the Bregman theorem to the three-dimensional matrices was made in 1987 by Dow and Gibson:

Theorem 14 [7]. *Let A be a three-dimensional $(0,1)$-matrix of order n and let the number of ones in the ith hyperplane of A be equal to r_i. Then*

$$\mathrm{per} A \le \prod_{i=1}^{n} r_i!^{1/r_i}.$$

Note that if all sums r_i in the hyperplanes of a matrix A are at most n and there exists a two-dimensional matrix B of order n with row sums r_i at which there is equality in the Bregman theorem then there exists a three-dimensional matrix at which the bound of Theorem 14 is attained. For constructing it, it suffices to distribute the rows of B over one-dimensional planes diagonally. If the sums in the hyperplanes of A are greater than n then Theorem 14 gives a rough estimate. For example, the permanent of the unit three-dimensional matrix E_n^3 is equal to $n!^2$, while Theorem 14 estimates it by $(n^2)!^{1/n}$.

At present, there is no good generalization of the Bregman theorem for the matrices whose number of ones in hyperplanes is close to the maximal number. As a candidate for such an estimate, we propose

Conjecture 1. *Let A be a d-dimensional $(0,1)$-matrix of order n. Suppose that the ith hyperplane of A contains r_i ones. Then*

$$\mathrm{per} A \le n!^{d-2} \prod_{i=1}^{n} \left\lceil \frac{r_i}{n^{d-2}} \right\rceil !^{1/\lceil r_i/n^{d-2} \rceil}.$$

Equality is attained at the matrices whose all two-dimensional planes of a fixed direction are equal to the two-dimensional matrix with r_i/n^{d-2} ones in the ith row giving equality in the Bregman theorem. In [41], it was proved that Conjecture 1 holds asymptotically for matrices with sufficiently large number of ones in hyperplanes.

Theorem 15. *Suppose that, for some $d \in \mathbb{N}$ and each $n \in \mathbb{N}$, we have a collection of n naturals $\{r_i(n)\}_{i=1}^n$ such that*

$$\min_{i=1,\dots,n} r_i(n)/n^{d-2} \to \infty \quad \textit{as } n \to \infty.$$

Denote by $\Lambda^d(n,r)$ the set of all d-dimensional $(0,1)$-matrices of order n for which the number of ones in the ith hyperplane is at most $r_i(n)$ and introduce the function $F(x) = \lceil x \rceil !^{1/\lceil x \rceil}$. Then

$$\max_{A \in \Lambda^d(n,r)} \mathrm{per} A \le n!^{d-2} e^{o(n)} \prod_{i=1}^{n} F\left(\frac{r_i(n)}{n^{d-2}}\right) \quad \textit{as } n \to \infty.$$

If Conjecture 1 held then it would be possible to estimate the permanent exactly enough by the sums of entries in the planes of each dimension. Indeed, suppose that A is a d-dimensional $(0,1)$-matrix of order n, l_β are the k-dimensional planes of a fixed dimension in A, $1 \le k \le d-1$, and r_β is the sum of entries in l_β. By Property 8 of the permanent of multidimensional matrices,

$$\mathrm{per} A = \sum_{\sigma \in \Sigma} \mathrm{per} A_\sigma,$$

where Σ is the set of mappings $\sigma : \{1, \dots, n\} \to I_n^{d-k}$ such that $(\sigma(1), \dots, \sigma(n))$ is the diagonal in a $(d-k)$-dimensional matrix and A_σ is the $(k+1)$-dimensional matrix of order n whose ith hyperplane is the plane $l_{\sigma(i)}$ of A. Note that the sum of entries in the hyperplanes of A_σ is $r_{\sigma(i)}$. Assuming that Conjecture 1 holds, we have

$$\mathrm{per}A_\sigma \le n!^{k-1} \prod_{i=1}^{n} h_{\sigma(i)}!^{1/h_{\sigma(i)}},$$

where $h_{\sigma(i)}$ stands for $\lceil r_{\sigma(i)}/n^{k-1} \rceil$. Then

$$\mathrm{per}A \le n!^{k-1} \sum_{\sigma \in \Sigma} \prod_{i=1}^{n} h_{\sigma(i)}!^{1/h_{\sigma(i)}}.$$

Defining the $(d-k)$-dimensional matrix B of order n with entries $b_\beta = h_\beta!^{1/h_\beta}$, $\beta \in I_n^{d-k}$, we can rewrite this inequality as

$$\mathrm{per}A \le n!^{k-1} \mathrm{per}B.$$

If the planes l_β are one-dimensional in these considerations then A_σ are two-dimensional, and their permanents can be estimated by means of the Bregman theorem. So we have

Proposition 3 [41]. *Suppose that A is a d-dimensional $(0, 1)$-matrix of order n, l_β are one-dimensional planes of A of some direction, and r_β is the sum of the entries in l_β. Construct the $(d-1)$-dimensional matrix B of order n with entries $b_\beta = r_\beta!^{1/r_\beta}$. Then*

$$\mathrm{per}A \le \mathrm{per}B.$$

If, in each hyperplane of B, we replace all entries by the maximal element in the hyperplane and use the linearity of the permanent then, on the right-hand side of the inequality in Proposition 3, there appears the permanent of some $(d-1)$-dimensional $(0, 1)$-matrix. For estimating this permanent we can apply Proposition 3, which gives the following result:

Theorem 16 [41]. *Let A be a d-dimensional $(0, 1)$-matrix of order n. Choose in A a system $\Theta_1, \dots, \Theta_{d-1}$ of embedded plane directions: the vector Θ_k defines a direction of k-dimensional planes, Θ_{k-1} is obtained from Θ_k by replacing one 0 by 1 (one of the variable coordinates becomes fixed). Assume that the hyperplanes of direction Θ_{d-1} are enumerated by numbers from 1 to n.*

Suppose that for $1 \le k \le d-1$ the unit entries of the k-dimensional planes of direction Θ_k lying in the ith hyperplane can be covered by $s_{i,k}$ $(k-1)$-dimensional planes of direction Θ_{k-1}. Then

$$\mathrm{per}A \le \prod_{k=1}^{d-1} \prod_{i=1}^{n} s_{i,k}!^{1/s_{i,k}}.$$

Equality in this inequality is attained, for example, for block-diagonal matrices. At the same time, the bound depends heavily on the location of ones in planes of the matrix and is often rough.

Theorem 16 can be used for estimating from above the permanents of matrices containing few ones and having a sufficiently regular structure. For example, consider an application of this theorem to the matrix whose permanent expresses the number of Latin squares.

Let $H_{LS}(n)$ be the hypergraph whose vertices are those of the cube $\mathbb{Z}_n^3$ and whose hyperedges are the MDS codes with distance 2 in $\mathbb{Z}_n^3$. Let $A_{LS}(n)$ be the adjacency matrix of parts of this hypergraph. As shown above, the dimension and order of $A_{LS}(n)$ are equal to n and its permanent coincides with the number of Latin squares of order n. For brevity, denote this number by $LS(n)$.

Note that, for each $2 \le k \le d-1$, for covering all ones in a k-dimensional plane of $A_{LS}(n)$, we need at most $k-1$ $(k-1)$-dimensional planes, and every one-dimensional plane of $A_{LS}(n)$ contains at most one 1. By Theorem 16,

$$LS(n) = n!\,\mathrm{per}A_{LS}(n) \le n! \prod_{k=1}^{n-1} \prod_{i=1}^{n} k!^{1/k} = \prod_{k=1}^{n} k!^{n/k},$$

which coincides with the familiar bound for the number of Latin squares.

In conclusion, compare the bounds given in this section on the following example: Let

$$A = \left(\begin{array}{cccc} 1&1&1&1\\ 1&1&0&0\\ 1&0&0&0\\ 1&0&0&0 \end{array} \times \begin{array}{cccc} 0&0&1&1\\ 1&1&1&1\\ 0&0&1&0\\ 0&0&1&0 \end{array} \times \begin{array}{cccc} 0&1&0&0\\ 0&1&0&0\\ 1&1&1&1\\ 1&1&0&0 \end{array} \times \begin{array}{cccc} 0&0&0&1\\ 0&0&0&1\\ 0&0&1&1\\ 1&1&1&1 \end{array} \right).$$

The permanent of A is 74, and the sum of the entries in each hyperplane is 8. Construct the matrix B with entries $b_{i,j} = s_{i,j}!^{1/s_{i,j}}$, where $s_{i,j}$ is the sum of the entries in the (i,j)th row of A. It can be check that $\mathrm{per}B \approx 104.23$, which is greater than the quantity $4! \cdot 2^{4/2} = 96$ of Conjecture 1. At the same time, Theorem 16 bounds the permanent of A by $4!^2 = 576$.

4.2. Upper Bounds for the Permanents of Polystochastic Matrices

For doubly stochastic matrices, a trivial upper bound for the permanent is given by Theorem 13, equals 1, and is attained at a diagonal matrix. We can see that an attempt to apply the results of Section 4.1 to the permanent of polystochastic matrices gives results very far from being true already for small dimensions. However, more exact bounds for the polystochastic matrices of given dimension and given order have not been found yet.

Nevertheless, for the set of polystochastic matrices of fixed dimension and arbitrarily large order, it was proved that the maximum of the permanent is close to the permanent of the uniform matrix. For obtaining this result, we have to go beyond the class of polystochastic matrices.

Denote by $M_n^d(\gamma)$ the set of nonnegative d-dimensional matrices of order n for which the sum of all entries is equal to γn, and the sum of the entries in each one-dimensional plane is at most 1. Note that the set Ω_n^d of all d-dimensional polystochastic matrices of order n coincides with $M_n^d(n^{d-2})$.

Consider the maximum function of the permanent on $M_n^d(\gamma)$:

$$P_n^d(\gamma) = \max_{A \in M_n^d(\gamma)} \mathrm{per}A.$$

The function $P_n^d(\gamma)$ is well defined since $M_n^d(\gamma)$ is compact.

It is not hard to check that, for two-dimensional matrices, $P_n^d(\gamma)$ is equal to γ^n. We may expect by analogy that $P_n^d(\gamma)$ grows monotonically with γ and for matrices of arbitrary dimension. But $P_n^d(\gamma)$ is in fact just piecewise monotone, which can be seen already by the example of three-dimensional matrices of order 2.

Despite the fact that the monotonicity of the maximum of the permanent of matrices in $M_n^d(\gamma)$ substantially complicates the proof of the upper bound, the author obtained

Theorem 17 [9]. *Let $d \ge 3$. For $\delta \in (0,1]$, the maximum of the permanent of d-dimensional matrices in $M_n^d(\gamma)$ for $\gamma = n^{d-3+\delta}$ and $n \to \infty$ is equal to*

$$P_n^d(\gamma) = \left((1+o(1)) \frac{\gamma}{e^{d-1}} \right)^n.$$

Note that, under the conditions of Theorem 17, the maximum of the permanent of matrices in $M_n^d(\gamma)$ asymptotically equals to the permanent of the matrices

$$\frac{\gamma}{n^{d-2}} J_n^d$$

with each entry equal to γ/n^{d-1}, whose asymptotic behavior is easily found by the Stirling formula.

Since for $\gamma = n^{d-2}$ the set $M_n^d(\gamma)$ coincides with the set of polystochastic matrices, their permanent satisfies

Theorem 18 [9]. *Let $d \geq 3$ and let Ω_n^d be the set of all d-dimensional polystochastic matrices of order n. Then*

$$\max_{A \in \Omega_n^d} \operatorname{per} A = \left((1 + o(1)) \frac{n^{d-2}}{e^{d-1}} \right)^n \quad \textit{as} \quad n \to \infty.$$

5. POLYSTOCHASTIC MATRICES, THEIR PROPERTIES AND GENERALIZATIONS

Since every convex sum of polystochastic matrices gives a polystoschastic matrix, all d-dimensional polystochastic matrices of order n constitute a convex polytope in the n^d-dimensional space. By analogy with the two-dimensional case, refer to the polytope of polystochastic matrices as the *Birkhoff polytope*. It is not hard to check that the dimension of the Birkhoff polytope is equal to $(n-1)^d$.

Recall that, in the class of doubly stochastic matrices, there exists a matrix unique up to a permutation of rows and columns consisting of 0s and 1s. In the d-dimensional case, for sufficiently large order n, there are many nonequivalent polystochastic $(0,1)$-matrices. Moreover, the positions of unities in polystochastic $(0,1)$-matrices constitute an MDS code with distance 2 in $\mathbb{Z}_n^d$. In this section, we refer to all polystochastic $(0,1)$-matrices as MDS *codes*.

Perhaps the easiest example of an MDS code is given by the d-dimensional matrix of order n such that its entry m_α is equal to 1 if and only if $\alpha_1 + \cdots + \alpha_d \equiv 0 \bmod n$. In what follows, we denote such a matrix by $\mathcal{M}_n^d$.

5.1. Constructions of Polystochastic Matrices

Let us give several ways of constructing polystochastic matrices.

Firstly, a polystochastic matrix can be obtained as a convex sum of other polystochastic matrices.

Somewhat more difficult way is the construction by the Kronecker product. Let A be a d-dimensional matrix of order n_1 and let B be a d-dimensional matrix of order n_2. Refer as the *Kronecker product* $A \otimes B$ of matrices A and B to the d-dimensional matrix C of order $n_1 n_2$ such that

$$c_\gamma = a_\alpha b_\beta \quad \text{for} \quad \gamma = n_2 * (\alpha - 1) + \beta, \quad \alpha \in I_{n_1}^d, \quad \beta \in I_{n_2}^d,$$

where addition and multiplications are carried out coordinatewise. It is not hard that the Kronecker product of two polystochastic matrices is a polystochastic matrix.

The following method makes it possible to obtain polystochastic matrices of larger dimension from matrices of smaller dimension: Let A be a d-dimensional polystochastic matrix of order n. Construct a $(d+1)$-dimensional matrix B of the same order. Fill one hyperplane Γ in B by entries of A. The remaining entries in one-dimensional planes perpendicular to the chosen hyperplane are filled uniformly: put the element $b_\beta \notin \Gamma$ to be equal to $(1 - a_\alpha)/(n-1)$, where α is the projection of the index β to Γ.

The last method of constructing polystochastic matrices has as the source the Marcus–Minc conjecture for doubly stochastic matrices. In [42], they assumed that the permanent of every doubly stochastic matrix A of order n is at least the permanent of the matrix

$$B = \frac{1}{n-1} \cdot (n J_n^2 - A).$$

If this conjecture is true then, using it, we can obtain one more proof of the van der Waerden conjecture.

For a d-dimensional polystochastic matrix A of order n, the matrix $B = 1/(n-1)\cdot(nJ_n^d - A)$ is also polystochastic. But already for the matrices of dimensions 3 and 4, the generalization of the Marcus–Minc conjecture fails. For the three-dimensional matrices, it suffices to consider the counterexample of the matrix $\mathcal{M}_3^3$ and, for four-dimensional matrices, the matrix that is obtained by applying the previous construction to $\mathcal{M}_3^3$.

5.2. Polystochastic Matrices and Classical Theorems for Two-Dimensional Matrices

Although the permanent of every doubly stochastic matrix is positive, among polystochastic matrices there exist matrices whose permanent is zero. The easiest example of a polystochastic matrix with zero permanent is the MDS code $\mathcal{M}_n^d$ of odd dimension d and even order n. With the use of the procedure described in Example 4 of calculating the sum of all indices over an unit diagonal, it is easy to check that the permanent of $\mathcal{M}_n^d$ is zero.

But no polystochastic matrix of dimension 4 or of any other even dimension with zero permanent has been found. Therefore, we can formulate the following conjecture, which is very important for the theory of multidimensional permanents:

Conjecture 2. *All polystochastic matrices of even dimension have positive permanent.*

Although this assertion is easy to prove for doubly stochastic matrices, all attempts to generalize the available proofs to the multidimensional case have been unsuccessful.

No polystochastic matrix of odd order with zero permanent has been found either. This makes it possible to propose

Conjecture 3. *All polystochastic matrices of odd order have positive permanent.*

In confirmation of this conjecture, we can prove the following

Theorem 19. *Let A be a d-dimensional matrix of order* 3. *Then the permanent of A is greater than zero.*

Proof. Note that, for each dimension, an MDS code of order 3 is unique up to equivalence and has nonzero permanent. The exact value of its permanent is given in Section 6.

Suppose that A is a polystochastic matrix of order 3 with zero permanent and assume that one of its one-dimensional planes contains at least two nonzero entries. We may assume without loss of generality that $a_{\alpha^1} \neq 0$ and $a_{\alpha^2} \neq 0$, where $\alpha^1 = (1, 3, \dots, 3)$ and $\alpha^2 = (2, 3, \dots, 3)$.

Denote by A^3 the hyperplane of the form $(3, *, \dots, *)$, where $*$ takes any three values from 1 to 3; A^3 is a $(d-1)$-dimensional polystochastic matrix of order 3. Denote by B the minor of the entry $a_{3,\dots,3}$ in A^3 and prove that B is a $(d-1)$-dimensional matrix of order 2 consisting of 0s and 1s.

In B choose a nonzero entry a_β, where $\beta = (3, x_2, \dots, x_d)$ and $x_i \in \{1, 2\}$, and denote by $a_{\overline{\beta}}$ the diagonal entry to a_β in B: $\overline{\beta} = (3, \overline{x}_2, \dots, \overline{x}_d)$, $\overline{x}_i = 3 - x_i$. Since we have assumed that the permanent of A is zero, the entry a_{β^1}, $\beta^1 = (1, \overline{x}_2, \dots, \overline{x}_d)$, complementing a_{α^2} and a_β to a diagonal is zero. Similarly, the entry a_{β^2} with $\beta^2 = (2, \overline{x}_2, \dots, \overline{x}_d)$ is zero as the element complementing a_{α^1} and a_β to a diagonal.

Note that the one-dimensional plane defined by the vector $(*, \overline{x}_2, \dots, \overline{x}_d)$, contains two zero entries a_{β^1} and a_{β^2}; and so the remaining entry $a_{\overline{\beta}}$ is 1. Similarly, taking $a_{\overline{\beta}}$ as the initial entry in the submatrix B, we infer that a_β is equal to 1.

Thus, the minor B of the element $a_{3,\dots,3}$ in the hyperplane A^3 is a $(0, 1)$-matrix. Then the hyperplane A^3 in A is also a $(0, 1)$-matrix because if some minor of a polystochastic matrix is a $(0, 1)$-matrix then the whole matrix is filled by 0s and 1s.

It remains to observe that if, in a polystochastic matrix A of order 3, some hyperplane is a $(0, 1)$-matrix then A is a convex sum of two MDS codes, and hence its permanent is positive.

Theorem 19 is proved. □

As was observed above, not all polystochastic matrices have positive permanent, and hence even fewer such matrices can be expended into the sum of diagonal matrices. For example, using the Latin squares with nonzero number of transversals containing the elements occurring in no transversal, we can construct a three-dimensional polystochastic matrix with positive permanent not representable as a sum of diagonal matrices. The existence of these Latin squares was proved in [43].

Nevertheless, among polystochastic matrices of even dimension, no examples have been found yet that could not be decomposed into a sum of diagonal matrices. Therefore, we suggest that Conjecture 2 about the positivity of the permanent can be strengthened to the following assertion that generalizes the Birkhoff theorem:

Conjecture 4. *All polystochastic matrices of even dimension can be represented as a nonnegative linear combination of diagonal matrices.*

The Birkhoff theorem has yet another interpretation which states that the angles of the Birkhoff polytope of doubly stochastic matrices are exactly the diagonal matrices. It is not hard to see that every MDS code is a vertex in the Birkhoff polytope of polystochastic matrices but for a dimension greater than two, all angular matrices are not exhausted by MDS codes. The easiest example of such a vertex of the Birkhoff polytope was found in [44] and [45]:

$$\left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{matrix} \times \begin{matrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{matrix} \times \begin{matrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{matrix}\right).$$

For matrices of dimension $d \geq 3$ and order $n \geq 4$, the vertices of the Birkhoff polytope that are not MDS codes can be constructed by means of Latin parallelepipeds not complementable to Latin hypercubes.

The exact number of vertices in the Birkhoff polytope of polystochastic matrices and also the ratio of MDS codes to all vertices remain unknown. The only result in this area is the construction of vertices of the polytope of three-dimensional polystochastic matrices proposed by Linial and Luria in [44].

Theorem 20. *The number of vertices in the Birkhoff polytope of three-dimensional polystochastic matrices is at least $M(n)^{3/2-o(1)}$, where $M(n)$ is the number of* MDS *codes of order n.*

Note that if, for the polytope of polystochastic matrices of even dimension, it were possible to show that all its vertices have positive permanent then Conjecture 2 would hold.

In the rest of the section, we consider the problem of minimum and maximum of the permanent for polystochastic matrices.

Recall that, in the two-dimensional case, the maximum of the permanent is attained at diagonal matrices; and, by the van der Waerden conjecture (Theorem 12), the minimum is attained at the uniform matrix. For polystochastic matrices, no matrices at which the minimum and the maximum of the permanent would be attained have been found yet. However, as in the two-dimensional case, the uniform matrix is a local extremum for the permanent on the set of polystochastic matrices though the sign of this extremum depends on the dimension:

Theorem 21 [9]. *The uniform matrix J_n^d is a local extremum for the permanent on the set of d-dimensional polystochastic matrices of order n: if d is even then J_n^d is a local minimum of the permanent, and if d is odd then it is a local maximum.*

We can verify the absence of a global extremum of the permanent at the uniform matrix by means of the following examples:

For order $n \leq 10$, we can find a three-dimensional MDS code whose permanent is greater than that of the uniform matrix. For the four-dimensional matrices of order $n \leq 6$, the permanent of the MDS code $\mathcal{M}_n^4$ is greater not only than that of the uniform matrix but also than the permanent of any other MDS code. Moreover, after considering a number of different polystochastic matrices, we may surmise that there holds

Conjecture 5. *The maximum of the permanent of polystochastic matrices is attained at an MDS code.*

Recall that if Conjecture 2 is true then the minimum of the permanent of polystochastic matrices of even dimension is different from zero. By analogy with the van der Waerden conjecture, as the most probable candidate for a matrix with minimal permanent, it stands to reason to consider the uniform matrix J_n^d, especially because the permanent has a local minimum at this matrix in even dimension. In [8], Dow and Gibson conjectured that, among the multidimensional matrices that are convex linear combinations of diagonal matrices, the permanent attains its minimum at a matrix proportional to the uniform matrix.

Both these conjectures are false since, by Theorem 21, for an odd dimension, the uniform matrix is a local maximum on the set of polystochastic matrices, and, for an even dimension, they are refuted by the following construction:

Proposition 4. *Suppose that the order n is odd or the dimension d is even, $d \geq 3$. Then there exists a polystochastic d-dimensional matrix H_n^d of order n whose permanent is asymptotically less than that of the uniform matrix:*

$$\mathrm{per} H_n^d \leq \left(\frac{1}{2} + o(1)\right)^n \mathrm{per} J_n^d.$$

Proof. Denote the MDS code $\mathcal{M}_n^d$ of order n by $\mathcal{M}$ and designate the MDS code whose entry m_α is equal to 1 if and only if $\sum_{i=1}^d \alpha_i \equiv 1 \bmod n$ as $\mathcal{M}'$. The matrices $\mathcal{M}$ and $\mathcal{M}'$ are equivalent, and their permanents coincide. Put

$$H_n^d = 1/2\,(\mathcal{M} + \mathcal{M}').$$

Note that every unit diagonal in $\mathcal{M}$ or $\mathcal{M}'$ becomes in H_n^d a diagonal filled by entries 1/2; therefore,

$$\mathrm{per} H_n^d \geq 2^{-n}(\mathrm{per}\mathcal{M} + \mathrm{per}\mathcal{M}') = 2^{-n+1}\mathrm{per}\mathcal{M}.$$

Show that the other diagonals do not influence the permanent of H_n^d. Suppose that, for some $k \in \{1, \dots, n-1\}$, there exists a diagonal $(\alpha^1, \dots, \alpha^n)$ passing through k unit entries of $\mathcal{M}'$ and $n-k$ unit entries of $\mathcal{M}$:

$$\sum_{i=1}^d \alpha_i^j \equiv 1 \bmod n \quad \text{for } j = 1, \dots, k, \qquad \sum_{i=1}^d \alpha_i^j \equiv 0 \bmod n \quad \text{for } j = k+1, \dots, n.$$

Put $S = \sum_{j=1}^n \sum_{i=1}^d \alpha_i^j$. As in Example 4, the different summation orders give

$$S \equiv k \bmod n, \qquad S \equiv d\,\frac{n(n+1)}{2} \bmod n.$$

If the order n is odd or the dimension d is even then the last equality takes the form $S \equiv 0 \bmod n$, and hence the matrix H_n^d does not contain new diagonals influencing the permanent compared to $\mathcal{M}$ and $\mathcal{M}'$:

$$\mathrm{per} H_n^d = 2^{-n+1}\mathrm{per}\mathcal{M}.$$

Recall that, by Theorem 18, the maximum of the permanent of polystochastic matrices is asymptotically equal to the permanent of the uniform matrix. Consequently, the permanent of the MDS code $\mathcal{M}$ satisfies

$$\mathrm{per}\mathcal{M} \leq (1 + o(1))^n \mathrm{per} J_n^d.$$

Thus,

$$\mathrm{per} H_n^d \leq \left(\frac{1}{2} + o(1)\right)^n \mathrm{per} J_n^d \quad \text{as} \quad n \to \infty.$$

Proposition 4 is proved. □

Table 1.

n	3	4	5	6	7	8
$\mathrm{per} H_n^4$	6.75	32	207.8125	1782	19574.734	273920
$\mathrm{per} J_n^4$	8	54	552.96	8000	155455.227	3906984.375

In fact, even for small orders n, the permanent of H_n^d is substantially less than that of J_n^d. We give a small Table 1 of the values of the permanents of these matrices in the four-dimensional case.

Moreover, calculating the number of transversals in Latin hypercubes of order 3 (which is done in Section 6), we can verify that the permanent of H_3^d is less than that of the uniform matrix J_3^d for all $d \geq 3$, which makes H_n^d a new candidate for a matrix with minimal permanent.

Note that, among polystochastic matrices of even dimension, there exists MDS codes whose permanent is less than that of the uniform matrix. An example of such a four-dimensional MDS code of order 6 will be given in the section on Latin cubes and hypercubes.

Basing on the above examples, we can make the following conjecture on the location of local extrema of the multidimensional permanent:

Conjecture 6. *All local extrema of the permanent in the Birkhoff polytope are located at the vertices or centers of its planes.*

5.3. The Permanents of MDS Codes

Since MDS codes are vertices of the Birkhoff polytope of polystochastic matrices, their study gives useful information about the permanents of the matrices from this polytope. Consider the properties of the permanents of some MDS codes.

Let $G = \langle \{g_1, \dots, g_n\}, *\rangle$ be a commutative group of order n. Refer as a *commutative d-dimensional MDS code* of order n to a $(0,1)$-matrix M whose entry m_α is equal to one if and only if $g_{\alpha_1} * \dots * g_{\alpha_d} = e$, where e is the identity element of G. Note that every MDS code $M(g)$ constructed from the relation $g_{\alpha_1} * \dots * g_{\alpha_d} = g$ for some $g \in G$ is equivalent to the MDS code M.

Proposition 5. *Every d-dimensional MDS code M of order n is transitive, and its permanent divides by n^{d-2}.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ be the indices of two unit entries in M. Then the transformation of coordinates $(x_1, \dots, x_d) \mapsto (y_1, \dots, y_d)$, defined by the relations $g_{x_i} * g_{\alpha_i}^{-1} * g_{\beta_i} = g_{y_i}$, $i = 1, \dots, d$, does not change M and takes the unit entry with index α to the entry with index β. Hence, M is a transitive matrix.

By Property 5, the permanent of every transitive matrix divides by the number of 1s in each of its hyperplanes. It only remains to observe that a hyperplane of M contains exactly n^{d-2} ones.

Proposition 5 is proved. □

The matrix $\mathcal{M}_n^d$ is the simplest example of a commutative MDS code. In Section 6, we estimate its permanent from below.

Theorem 22. *If the dimension d is odd and the order n is even then* $\mathrm{per}\mathcal{M}_n^d = 0$*; otherwise,* $\mathrm{per}\mathcal{M}_n^d \geq (nn!)^{\lfloor d/2 \rfloor - 1}$.

5.4. Other Generalizations of Stochasticity

The definition of polystochastic matrix we have adopted is not the only way of generalizing the concept of doubly polystochasticity. Therefore, we consider other possible generalizations and their properties.

Recall that a nonnegative multidimensional matrix is called *polystochastic* if the sum of the entries in its every one-dimensional plane is equal to 1. This definition distinguishes a rather narrow class in the set of all multidimensional matrices. As a wider class, we can consider the nonnegative d-dimensional matrices for which the sum of the entries is the same in every k-dimensional plane, $1 \le k \le d-1$. Call these matrices k-*stochastic*. The set of polystochastic matrices coincides with the set of 1-stochastic matrices for which the sum of the entries in each one-dimensional plane is equal to 1 and embeds in the set of k-stochastic matrices for which the sum of the entries in the k-dimensional planes is n^{k-1}. Some easy properties of k-stochastic matrices were proved by Jurkat and Ryser in [25].

The set of k-stochastic matrices obviously forms a convex polytope in the space of all multidimensional matrices, and, for each k from 1 to $d-1$, we can try to describe all vertices of this polytope and estimate their number. For example, by Brualdi and Csima in [46], the most complete description was carried out of the vertices of the polytope of three-dimensional matrices for which the sum of the entries in every hyperplane is equal to 1; and the constructions they proposed give a good lower bound for the number of vertices.

The extension of the class of polystochastic matrices to the k-stochastic matrices significantly influences the extrema of the permanent. Recall that, by Theorem 21, the uniform matrix is a local maximum for the permanent of polystochastic matrices of odd dimension and is a local minimum for matrices of even dimension. But already in the class of 2-stochastic matrices, the uniform matrix is not a local extremum.

Indeed, choose $\varepsilon > 0$ small enough and construct d-dimensional matrices K_2^d of order 2:

$$K_2^3 = \begin{pmatrix} \varepsilon & -\varepsilon & \times & 0 & 0 \\ 0 & 0 & & -\varepsilon & \varepsilon \end{pmatrix}, \qquad K_2^d = \left(K_2^{d-1} \times -K_2^{d-1}\right).$$

Complement the matrix K_2^d by zeros to a d-dimensional matrix K_n^d of order n. Note that the sum of the entries in every two-dimensional plane of K_n^d is zero, and hence the matrix $J_n^d + K_n^d$ is 2-stochastic. Decomposing its permanent over the hyperplane containing ε, we infer

$$\operatorname{per}\left(J_n^d + K_n^d\right) = \operatorname{per} J_n^d + (-1)^{d+1}\varepsilon^2 2^{d-2}\frac{(n-2)!^{d-1}}{n^{n-2}}.$$

Therefore, if d is even then $\operatorname{per}\left(J_n^d + K_n^d\right) < \operatorname{per} J_n^d$ and if d is odd then $\operatorname{per}\left(J_n^d + K_n^d\right) > \operatorname{per} J_n^d$. At the same time, Theorem 21 states the opposite.

The properties of the permanents of d-dimensional $(d-1)$-stochastic $(0,1)$-matrices of order n with l ones in each hyperplane were considered by Dow and Gibson [8]. They denoted the set of these matrices by $\Lambda(n,d,l)$. Let us cite some assertions from [8]:

Proposition 6. *If $A \in \Lambda(n,d,2)$ then the permanent of A is either* 0 *or* 2^t *for some* $1 \le t \le n/2$.

Proposition 7. *For every even n and $d \ge 3$, there exists a matrix $A \in \Lambda(n,d,n^{d-2})$ with zero permanent.*

Proposition 8. *For every odd n and $d \ge 3$, there exists a matrix $A \in \Lambda(n,d,m^{d-2})$ with zero permanent, where m is any even number satisfying the inequality*

$$m(1 + m^{-1/(d-1)}) \le n.$$

Let us now formulate in terms of multidimensional matrices the result on the weighted 1-factors of hypergraphs of [14], which makes it possible to estimate from below and above the permanent of some subclass of $(d-1)$-stochastic matrices.

Refer to a d-dimensional nonnegative matrix A of order n as λ-*balanced* if

$$a_\alpha / a_\beta \le \lambda$$

for every entries a_α and a_β. Obviously, if a matrix λ-balanced then all its entries are positive.

Theorem 23 [14]. *Let A be a d-dimensional λ-balanced $(d-1)$-stochastic matrix of order n. Then there exists a real number $\varkappa = \varkappa(d, \lambda) > 0$ such that*

$$n^{-\varkappa} e^{-n(d-1)} \leq \operatorname{per} A \leq n^{\varkappa} e^{-n(d-1)}.$$

Finally, as the most general case, we consider the class of matrices for which the sum of the entries in all planes of some dimension are known but different. The set of these matrices still constitutes a convex polytope in the space of all multidimensional matrices, and in [25] an estimate was obtained for the cardinality of the support of all its vertices in the three-dimensional case. It is easy to generalize this result to the case of an arbitrary dimension. For this we need

Lemma 1. *Consider the space of all d-dimensional matrices of order n for which the sums of the entries in all $(d-k)$-dimensional planes, $0 \leq k \leq d$, are known. The dimension of this space is equal to*

$$n^d - \sum_{i=0}^{k} C_d^i (n-1)^i.$$

Proof. The formula holds obviously for $k = d$. Choose an index $\alpha \in I_n^d$ and denote by $V_n^d(k, \alpha)$ the set of the indices of the entries lying in the union of all k-dimensional planes passing through α.

It is not hard to check that, in the space of d-dimensional matrices of order n with known sums in all $(d-k)$-dimensional planes over any assignment of entries whose indices do not lie in $V_n^d(k, \alpha)$, the values of all other entries of the matrix are always uniquely reconstructible. Therefore, the dimension of this space is $n^d - \left|V_n^d(k, \alpha)\right|$.

It only remains to observe that the cardinality of $V_n^d(k, \alpha)$ does not depend on the choice of α and equals $\sum_{i=0}^{k} C_d^i (n-1)^i$. Lemma 1 is proved. □

Given a d-dimensional matrix of order n, enumerate all k-dimensional planes by the numbers from 1 to t, where $t = C_d^k n^{d-k}$. Let $\mathfrak{M}_k(s_1, \ldots, s_t)$ denote the class of all nonnegative matrices for which the sum of the entries in the ith k-dimensional plane is equal to s_i. Refer to a matrix $A \in \mathfrak{M}_k(s_1, \ldots, s_t)$ as *k-vertex* if it cannot be expressed as a nontrivial linear combination of matrices of this class.

Recall that the support of a multidimensional matrix is the set of the indices of its nonzero entries and denote by $N(A)$ the cardinality of the support of A.

Proposition 9. *The cardinality of the support $N(A)$ of a d-dimensional k-vertex A of order n satisfies the inequality*

$$N(A) \leq \sum_{i=0}^{d-k} C_d^i (n-1)^i.$$

Proof. Let A be a d-dimensional matrix of order n of class $\mathfrak{M}_k(s_1, \ldots, s_t)$ and let $N = N(A)$ be the cardinality of the support of this matrix. Enumerate all elements of the support from 1 to N. Let U denote the incidence matrix of the support of A and k-dimensional planes. The matrix U is a rectangular two-dimensional $(0,1)$-matrix U of size $t \times N$. Note that each column of U contains exactly C_d^k ones. Moreover, the union of every n^{d-k} rows of U, corresponding to the set of k-dimensional planes of one direction, contains exactly one unit entry in each column.

It is not hard to see that A is a k-vertex if and only if the columns of U are linearly independent. Since the row and column rank coincide, for the linear independence of the columns of U it is necessary that their number be at most the number of linearly independent characteristic functions of k-dimensional planes in a d-dimensional matrix of order n. The class $\mathfrak{M}_k(s_1, \ldots, s_t)$ is defined by a system of t linear equations on the entries of a d-dimensional matrix of order n, and hence the number of linearly independent characteristic functions of k-dimensional planes is equal to the codimension of the space of d-dimensional matrices of order n for which the sums in all k-dimensional planes are known. Lemma 1 implies that this number equals $\sum_{i=0}^{d-k} C_d^i (n-1)^i$.

Proposition 9 is proved. □

6. MULTIDIMENSIONAL PERMANENTS AND LATIN HYPERCUBES

Recall that a *d-dimensional Latin hypercube Q of order n* is the d-dimensional matrix of order n whose all entries take n different values and all values in each of its one-dimensional planes are distinct. A two-dimensional Latin hypercube is called a *Latin square*, and a three-dimensional Latin hypercube is called a *Latin cube*. Two d-dimensional Latin hypercubes are *equivalent* if the corresponding $(d+1)$-dimensional matrices are equivalent. A *transversal* in a Latin hypercube is a diagonal containing all n different values. Let $T(Q)$ denote the number of transversals in Q.

Let $\mathcal{Q}_n^d$ denote the d-dimensional Latin hypercube of order n such that $q_{\alpha_1,\dots,\alpha_d} = \alpha_0$ whenever

$$\sum_{i=0}^{d} \alpha_i \equiv 0 \bmod n.$$

It is easy to see that to $\mathcal{Q}_n^d$ there corresponds the $(d+1)$-dimensional MDS code $\mathcal{M}_n^{d+1}$ of order n.

Each Latin hypercube can be regarded as the table of the values of some quasigroup. A *d-ary quasigroup of order n* is an algebraic system consisting of the set $\mathbb{Z}_n^d$ and a d-ary operation

$$f : \mathbb{Z}_n^d \to \mathbb{Z}_n, \qquad f(x_1,\dots,x_d) = x_0$$

that is bijectively invertible with respect to each variable. A *transversal in a quasigroup f* is a collection $\{(a_0^i,\dots,a_d^i)\}_{i=1}^n$ consisting of the values of the variables and the value of the quasigroup such that, for each $k \in \{0,\dots,d\}$, the entries a_k^i range over all n possible values.

It is not hard to see that to every d-ary quasigroup of order n there corresponds in a one-to-one fashion a d-dimensional Latin hypercube of the same order and the numbers of transversals in them coincide.

6.1. Application of the Multidimensional Permanent for Calculating the Number of Transversals

As was observed in Section 2, the number of transversals in a $(d-1)$-dimensional Latin hypercube is equal to the permanent of the corresponding d-dimensional MDS code. Consider a new way of relating the number of transversals with the multidimensional permanent and give its several applications.

Let Q be a d-dimensional Latin hypercube of order n and let $X = (x_1,\dots,x_n)$ be a set of variables. Construct from Q the hypercube $Q(X)$ with elements $\{x_i\}_{i=1}^n$ such that the element $q_\alpha(X)$ of $Q(X)$ is equal to x_i if and only if the element q_α of Q is equal to i. Obviously, the permanent of $Q(X)$ is a polynomial of degree n in the variables $x_1,\dots,x_n$. Refer to this polynomials as the *permanent* of Q.

Some properties of a Latin hypercube are reflected by its permanent [47]. We use the fact that the number of transversals $T(Q)$ in Q coincides with the coefficient at $\prod_{i=1}^n x_i$ in the permanent of the hypercube. This observation enables us to express the number of transversals in every Latin hypercube as the sum of the permanents of the matrices of special kind.

For a polynomial $p(x_1,\dots,x_n)$ of degree n of the variables $x_1,\dots,x_n$, introduce the function $\langle r\rangle p(X)$ equal to the sum of the values of $p(X)$ over all $(0,1)$-vectors of length n and weight r. By the inclusion-exclusion principle, the coefficient at $\prod_{i=1}^n x_i$ in $p(X)$ is $\sum_{r=0}^n (-1)^{n-r}\langle r\rangle p(X)$. Therefore, the number of transversals in Q satisfies the equality

$$T(Q) = \sum_{r=0}^{n} (-1)^{n-r}\langle r\rangle \operatorname{per} Q(X),$$

where $\langle r\rangle \operatorname{per} Q(X)$ can be regarded as the sum of the permanents of d-dimensional $(0,1)$-matrices of order n whose each one-dimensional plane contains exactly r ones.

The above-described correspondence makes it possible to prove that each line Latin hypercube of even order has evenly many transversals. Refer to a d-dimensional matrix Q of order n whose entries take exactly n values as the *line Latin hypercube* if there exists a direction of one-dimensional planes such that, in each one-dimensional plane of this direction, all entries are distinct. Since all Latin hypercubes are line Latin, we have

Theorem 24. *Each d-dimensional Latin hypercube of even order has evenly many transversals.*

The notion of line Latin squares was used by Balasubramanian in [48] for proving an analogous two-dimensional assertion. The proof of Theorem 24 bases on the following property of the multidimensional permanent, which is of importance in its own right:

Proposition 10. *Let $A = (A_1, \dots, A_n)$ be a d-dimensional integer matrix of order n, where A_i are hyperplanes of one direction of A. Let $A' = (A_1 + A_2, A_2, \dots, A_n)$ be the matrix obtained by the addition of two hyperplanes. Then the permanents of A and A' have the same parity.*

Proof. Since the permanent is a multilinear function of hyperplanes, we have

$$\mathrm{per} A' = \mathrm{per} A + \mathrm{per}(A_2, A_2, \dots, A_n).$$

To each diagonal $(\alpha^1, \alpha^2, \alpha^3 \dots, \alpha^n)$ in $(A_2, A_2, \dots, A_n)$, where the entry with index α^i lies in the hyperplane with number i, we can assign the diagonal $(\alpha^2, \alpha^1, \alpha^3 \dots, \alpha^n)$. The products of the entries of the matrix $(A_2, A_2, \dots, A_n)$ located on these diagonals are identical, and hence each such pair of diagonals makes an even contribution to the permanent. Proposition 10 is proved. □

The above-considered construction enables us to express through a linear combination of permanents of matrices of a special form not only the number of transversals in Latin hypercubes but also the permanent of every multidimensional matrix.

Proposition 11. *Let A be a d-dimensional matrix of order n. Let us construct from A a $(d-1)$-dimensional hypercube $Q(X)$ of order n of the variables $x_1, \dots, x_n$ as follows: To the entries $\{a_{\alpha_1,\dots,\alpha_{d-1},i}\}_{i=1}^n$ of a one-dimensional plane of A, assign the element*

$$q_{\alpha_1,\dots,\alpha_{d-1}} = \sum_{i=1}^{n} a_{\alpha_1,\dots,\alpha_{d-1},i} x_i$$

of the hypercube $Q(X)$. Then

$$\mathrm{per} A = \sum_{r=0}^{n} (-1)^{n-r} \langle r \rangle \mathrm{per}\, Q(X).$$

6.2. The Number of Transversals in Latin Hypercubes and the Permanent

Since the set of transversals in a $(d-1)$-dimensional Latin hypercube is equal to the permanent of certain d-dimensional MDS code, often results obtained for one of these objects are also applicable to the other. For example, using Theorem 18, one can estimate from above the number of transversals in Latin hypercubes [9]:

Theorem 25. *Let $T(d, n)$ be the maximal number of transversals in d-dimensional Latin hypercubes of order n. Then*

$$T(d, n) \le \left((1 + o(1)) \frac{n^{d-1}}{e^d} \right)^n \quad as \quad n \to \infty.$$

In particular, the number of transversals in Latin squares of order n is asymptotically at most

$$\left((1 + o(1))\, n/e^2 \right)^n.$$

In [49], some probabilistic construction was proposed that proves the asymptotic exactness of this estimate for Latin squares and allows us to construct an infinite series of Latin hypercubes attaining it. Moreover, [49] contains an alternative proof of Theorem 25.

The problem of estimating the maximal number of transversals in Latin squares $T(2, n)$ was posed by Wanless at the Loops'03 conference, and until recently the bound for $T(2, n)$ found in [50] has been the best:

$$b_1^n \le T(2, n) \le b_2^n \sqrt{n} n!,$$

where $b_1 \approx 1.719$ and $b_2 \approx 0.614$, which is equivalent to

$$T(2,n) \le \big((1+o(1))ne^{-c}\big)^n, \qquad c \approx 1.48.$$

Pass to the problem of a lower bound for the number of transversals in Latin hypercubes. Unfortunately, even for Latin squares, no good method was proposed for searching for a transversal when a square is given. Nevertheless, no Latin square of odd order without transversals has been found yet. The conjecture that all these squares contain a transversal are usually attributed to Ryser [51]. There have been several attempts to prove this conjecture but they all failed. Presently, the conjecture is verified by numerical experiments for Latin squares of order at most 9. More detailed information on transversals in Latin squares can be found in [52].

Return to the transversals in Latin hypercubes. Basing on an analysis of transversals in Latin hypercubes of small order and dimension, Wanless made the following conjectures in [52]:

Conjecture 7. *Each Latin hypercube of odd order has at least one transversal.*

Conjecture 8. *Each Latin hypercube of odd dimension has at least one transversal.*

Note that the first of these is a generalization of the Ryser conjecture to the multidimensional case, while the second is a relaxation of Conjecture 2 on the positivity of the permanent of polystochastic matrices to the case of MDS codes. For Latin hypercubes having a sufficiently good structure, it is possible not only to establish the existence of transversals but also estimate their number. For this, it will be convenient for us to regard a Latin hypercube as a quasigroup.

Let $f(x_1,\dots,x_k) = x_0$ and $g(y_1,\dots,y_l) = y_0$ be a k-ary quasigroup and an l-ary quasigroup of order n. Consider the $(k+l-1)$-ary quasigroup h of order n defined by the relation

$$f(x_1,\dots,x_{k-1}, g(y_1,\dots,y_l)) = x_0.$$

We refer to this quasigroup h as the *composition* of f and g.

Proposition 12. *If quasigroups f and g have $T(f)$ and $T(g)$ transversals then the number of transversals in h is at least $T(f)T(g)$:*

$$T(h) \ge T(f)T(g).$$

Proof. In f and g, choose the transversals $\big\{\big(a_0^i, a_1^i \dots a_{k-1}^i, i\big)\big\}_{i=1}^n$ and $\big\{\big(i, b_1^i \dots, b_l^i\big)\big\}_{i=1}^n$ respectively. Then the collection $\big\{\big(a_0^i,\, a_1^i,\dots,a_{k-1}^i, b_1^i,\dots,b_l^i\big)\big\}_{i=1}^n$ is a transversal in the quasigroup h. Proposition 12 is proved. □

Proposition 13. *Suppose that, for some $a \in \mathbb{Z}_n$, the quasigroup f' defined by the equation $f(x_1,\dots,x_{k-1},a) = x_0$ has $T(f')$ transversals and the quasigroup g' defined by $g(y_1,\dots,y_l) = a$ has $T(g')$ transversals. Then the composition of f and g has at least $n!T(f')T(g')$ transversals*:

$$T(h) \ge n!T(f')T(g').$$

Proof. Suppose that $\big\{\big(a_0^i,\dots,a_{k-1}^i\big)\big\}_{i=1}^n$ is a transversal in f', and let $\big\{\big(b_1^j,\dots,b_l^j\big)\big\}_{j=1}^n$ be a transversal in g'. Then, for each permutation $\sigma \in S_n$, the set

$$\big\{\big(a_0^i,\dots,a_{k-1}^i, b_1^{\sigma(i)},\dots,b_l^{\sigma(i)}\big)\big\}_{i=1}^n$$

is a transversal in h. Proposition 13 is proved. □

These assertions imply that the number of transversals in many completely reducible quasigroups is positive. Refer to a d-ary quasigroup h as *completely reducible* if it is the composition of some completely reducible quasigroups f and g; moreover, 1-ary and 2-ary quasigroups are assumed completely reducible.

Theorem 26. *Let h be a d-ary completely reducible quasigroup of order n and let d be odd. Then the corresponding d-dimensional Latin hypercube of order n contains at least $(nn!)^{(d-1)/2}$ transversals.*

If h is a d-ary completely reducible quasigroup of order n, d is even, and the most external quasigroup f in its composition has a transversal then the corresponding d-dimensional Latin hypercube of order n contains at least $(nn!)^{\lfloor (d-1)/2 \rfloor}$ transversals.

Note that, to the Latin hypercube $\mathcal{Q}_n^d$, there corresponds a completely reducible quasigroup. Therefore, we have

Theorem 27. *Suppose that the dimension d is even and the order n is even. Then $T(\mathcal{Q}_n^d) = 0$; otherwise, $T(\mathcal{Q}_n^d) \geq (nn!)^{\lfloor (d-1)/2 \rfloor}$.*

6.3. Transversals in Some Latin Hypercubes

Let us start the study of transversals in Latin cubes of a special form with the hypercube $\mathcal{Q}_n^d$.

The fact that each Latin hypercube $\mathcal{Q}_n^d$ of odd dimension d has at least one transversal was first obtained in [53]. The best lower bound on the number of transversals in this hypercube is given by Theorem 27, and the best upper bound is provided by Theorem 25.

Finding the asymptotics of the number of transversals $\mathcal{Q}_n^d$ is a difficult problem even for the Latin squares $\mathcal{Q}_n^2$. Vardi [54] conjectured that there exist constants $0 < c_1 < c_2 < 1$ such that the number of transversals in the squares $\mathcal{Q}_n^2$ meets the inequality

$$c_1^n n! \leq T(\mathcal{Q}_n^2) \leq c_2^n n! \qquad \text{for all odd } n \geq 3.$$

After some papers devoted to lower and upper bounds for $T(\mathcal{Q}_n^2)$, the Vardi Conjecture was strengthened by Wanless:

Conjecture 9 [52]. *Suppose that n is odd. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\frac{T(\mathcal{Q}_n^2)}{n!} \right) = -1.$$

Theorem 25 implies that this limit is at most -1, and [55] asserts that Conjecture 9 holds and announces an even more exact asymptotics for $T(\mathcal{Q}_n^2)$.

Conjecture 9 can be generalized to the multidimensional case on assuming that if the number of transversals in $\mathcal{Q}_n^d$ is nonzero then it is close the maximum:

Conjecture 10. *Suppose that n or d is odd. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\frac{T(\mathcal{Q}_n^d)}{n!^{d-1}} \right) = -1.$$

Consider how many transversals are contained in Latin hypercubes of small orders. For every dimension d, the Latin hypercube of order 2 is unique and coincides with $\mathcal{Q}_2^d$ up to equivalence. Recall that the number of transversals in $\mathcal{Q}_2^d$ is equal to $\operatorname{per} \mathcal{M}_2^{d+1}$. Using the formulas of Example 3 in Section 1, we conclude that $T(\mathcal{Q}_2^d) = 0$ for d even and $T(\mathcal{Q}_2^d) = 2^{d-1}$ for d odd.

For order 3, the d-dimensional Latin hypercube $\mathcal{Q}_3^d$ is also unique. For calculating the number of transversals in $\mathcal{Q}_3^d$, choose in it $(d-2)$-dimensional planes diagonally and compose of them $(d-1)$-dimensional hypercubes. Note that, in three of the six possible cases, the obtained hypercubes are Latin, and in the other half of the cases, the hyperplanes of the hypercubes are filled identically. Thus, the following recurrence holds:

$$T(\mathcal{Q}_3^d) = 3T(\mathcal{Q}_3^{d-1}) + 18T(\mathcal{Q}_3^{d-2}), \qquad T(\mathcal{Q}_3^2) = 3, \qquad T(\mathcal{Q}_3^3) = 27.$$

Solving this, we obtain

$$T(\mathcal{Q}_3^d) = 3^{d-2}(2^d - (-1)^d),$$

which is the sequence A080424 in [56].

The number of nonequivalent d-dimensional Latin hypercubes of order 4 grows exponentially with d. It is unknown yet if there exist other Latin hypercubes of order 4 in addition to $\mathcal{Q}_4^d$ with even d not having transversals. Using Propositions 12 and 13 of Section 5 and the theorem by Krotov and Potapov on the characterization of quasigroups of order 4 [57], we obtain

Theorem 28. *Each Latin hypercube of order* 4 *and of odd dimension has a transversal.*

In conclusion, we expose some results on the number of transversals in Latin hypercubes of small dimensions and orders. As the material for this part, we used the hypercubes listed by McKay and Wanless [58].

There exist five nonequivalent Latin cubes of order 4. The maximal number of transversals is contained in the cubes

$$\begin{array}{cccc} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{array} \times \begin{array}{cccc} 1&0&3&2\\ 0&1&2&3\\ 3&2&1&0\\ 2&3&0&1 \end{array} \times \begin{array}{cccc} 2&3&0&1\\ 3&2&1&0\\ 0&1&2&3\\ 1&0&3&2 \end{array} \times \begin{array}{cccc} 3&2&1&0\\ 2&3&0&1\\ 1&0&3&2\\ 0&1&2&3 \end{array},$$

$$\begin{array}{cccc} 0&1&2&3\\ 1&0&3&2\\ 2&3&1&0\\ 3&2&0&1 \end{array} \times \begin{array}{cccc} 1&0&3&2\\ 0&1&2&3\\ 3&2&0&1\\ 2&3&1&0 \end{array} \times \begin{array}{cccc} 2&3&1&0\\ 3&2&0&1\\ 1&0&3&2\\ 0&1&2&3 \end{array} \times \begin{array}{cccc} 3&2&0&1\\ 2&3&1&0\\ 0&1&2&3\\ 1&0&3&2 \end{array}$$

and is equal to 256; moreover, the last is equivalent to the cube $\mathcal{Q}_4^3$.

The minimal number of transversals is equal to 96 and contained in the cube

$$\begin{array}{cccc} 0&1&2&3\\ 1&0&3&2\\ 2&3&1&0\\ 3&2&0&1 \end{array} \times \begin{array}{cccc} 1&0&3&2\\ 2&3&1&0\\ 3&2&0&1\\ 0&1&2&3 \end{array} \times \begin{array}{cccc} 2&3&1&0\\ 3&2&0&1\\ 0&1&2&3\\ 1&0&3&2 \end{array} \times \begin{array}{cccc} 3&2&0&1\\ 0&1&2&3\\ 1&0&3&2\\ 2&3&1&0 \end{array}.$$

Two remaining Latin cubes of order 4 contain 208 and 128 transversals respectively.

For order 5, there exist 15 nonequivalent Latin cubes, and they all have different numbers of transversals. The maximal number of transversals, equal to 3325, is contained in $\mathcal{Q}_5^3$, and the minimal number of transversals is 859. They also include Latin cubes having an even number of transversals, and hence in the multidimensional case, as well as in the two-dimensional case, the oddity of the order of a Latin hypercube does not imply the oddity of the number of its transversals.

There are already more than 200000 nonequivalent Latin cubes of order 6. The maximal number of transversals (57024) is contained in $\mathcal{Q}_6^3$, while the minimal number of transversals (7632) is attained at the two Latin cubes one of which has the following form:

$$\begin{array}{cccccc} \mathit{0}&\mathit{1}&\mathit{2}&\mathit{3}&\mathit{4}&\mathit{5}\\ \mathit{1}&\mathit{2}&\mathit{0}&\mathit{4}&\mathit{5}&\mathit{3}\\ \mathit{2}&\mathit{0}&\mathit{1}&\mathit{5}&\mathit{3}&\mathit{4}\\ \mathit{3}&\mathit{4}&\mathit{5}&\mathit{0}&\mathit{1}&\mathit{2}\\ \mathit{4}&\mathit{5}&\mathit{3}&\mathit{1}&\mathit{2}&\mathit{0}\\ \mathit{5}&\mathit{3}&\mathit{4}&\mathit{2}&\mathit{0}&\mathit{1} \end{array}, \quad \text{where} \quad \begin{array}{l} \mathit{0} = 0\,1\,2\,3\,4\,5\\ \mathit{1} = 1\,0\,3\,2\,5\,4\\ \mathit{2} = 2\,3\,4\,5\,0\,1\\ \mathit{3} = 3\,2\,5\,4\,1\,0\\ \mathit{4} = 4\,5\,0\,1\,2\,3\\ \mathit{5} = 5\,4\,1\,0\,3\,2 \end{array}.$$

We can check that the structure of this cube resembles that of the cube of order 4 with the minimal number of transversals. Observe also that the four-dimensional MDS code of order 6 corresponding to this Latin cube has the permanent less than the permanent of the uniform matrix J_6^4 and is a counterexample to the trivial generalization of the van der Waerden Conjecture to the multidimensional case.

Now, consider the four-dimensional Latin hypercubes. The number of nonequivalent 4-dimensional Latin hypercubes of order 4 is 26. The only hypercube of these, namely, $\mathcal{Q}_4^4$, has no transversals. The maximal number of transversals (5120) is contained in the hypercube

$$\begin{matrix} \mathit{0} & \mathit{1} & \mathit{2} & \mathit{3} \\ \mathit{1} & \mathit{0} & \mathit{3} & \mathit{2} \\ \mathit{2} & \mathit{3} & \mathit{0} & \mathit{1} \\ \mathit{3} & \mathit{2} & \mathit{1} & \mathit{0} \end{matrix},$$

where

$$\mathit{0} = \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}, \qquad \mathit{1} = \begin{matrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{matrix}, \qquad \mathit{2} = \begin{matrix} 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{matrix}, \qquad \mathit{3} = \begin{matrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \end{matrix}.$$

For order 5, there exist 86 four-dimensional Latin hypercubes, and they all contain transversals. The minimal number of their transversals is 60843, and they are contained in a hypercube to which there corresponds some reducible quasigroup; and the maximal number (321375 transversals) is contained in $\mathcal{Q}_5^4$.

7. CONDITIONS FOR POSITIVITY OF THE MULTIDIMENSIONAL PERMANENT

Recall that, in the two-dimensional case, the characterization of matrices with positive permanent is given by the König–Frobenius theorem (Theorem 8). As a possible generalization of this theorem to multidimensional matrices, we can propose the following assertion:

Conjecture 11. *Let A be a nonnegative matrix of even dimension. If the size $x_1 \times \cdots \times x_d$ of each nonzero submatrix in A satisfies*

$$\max_{i \neq j}(x_i + x_j) \leq n$$

then the permanent of A is positive.

Since a two-dimensional plane of a polystochastic matrix is a doubly stochastic matrix, all polystochastic matrices satisfy the conditions of this conjecture. For matrices of odd dimension, the analog of Conjecture 11 fails because of the existence of polystochastic matrices with zero permanent.

Another possible approach to characterizing the multidimensional matrices with positive permanent consists in describing the extremal matrices. Call a $(0, 1)$-matrix *extremal* if its permanent is equal to zero and, after changing any zero entry by one, the permanent increases. Refer to a multidimensional $(0, 1)$-matrix as *maximal* if, after changing any zero entry by one, its permanent increases, and as *minimal* if the change of any unit entry by zero decreases the permanent. Note that an extremal matrix is a maximal matrix with zero permanent.

As a particular case of Conjecture 4, we may assume that there holds

Conjecture 12. *Let A be a d-dimensional MDS code of order n, where d is even. Then A is a minimal matrix.*

As was shown in Section 2, the permanent of every d-dimensional $(0,1)$-matrix coincides with the number of 1-factors of some d-partite balanced hypergraph, and hence the conditions guaranteeing the existence of a 1-factor of such a hypergraph can be reformulated as the conditions of the positivity of the permanent. One of these conditions is given in [13]:

Theorem 29. *Let A be a d-dimensional $(0,1)$-matrix of order n. If there exist two directions of one-dimensional planes such that all planes of one direction contain strictly greater than $n/2$ ones and the planes of the second direction contain at least $n/2$ ones then the permanent of A is greater than zero.*

Note that, in this theorem, the condition of the presence of a one-dimensional planes that is at least half composed of 1s cannot be relaxed since there exist multidimensional $(0,1)$-matrices whose each one-dimensional plane is half filled with 1s (Example 4).

Many conditions for the existence of 1-factors of hypergraphs are asymptotic and work only if the hypergraph has sufficiently many vertices. With their help, certain conditions can be obtained for the positivity of the permanent of matrices of sufficiently large order. For example, one of the results in [59] can be reformulated as

Theorem 30. *Introduce the function*

$$t(n) = \begin{cases} n^2 - \big(n - \lfloor (n-1)/3 \rfloor\big)\big(n - \lfloor n/3 \rfloor\big) + 1, & \textit{if} \quad n \not\equiv 2 \bmod 3, \\ n^2 - \big(n - \lfloor (n-2)/3 \rfloor\big)^2 + 2, & \textit{if} \quad n \equiv 2 \bmod 3. \end{cases}$$

Let A be a nonnegative three-dimensional matrix of order n whose each hyperplane contains at least $t(n)$ nonzero entries. Then, starting from certain n, the permanent of A is greater than zero. Moreover, if the condition on the number of positive entries in hyperplanes fails then there is a matrix with zero permanent.

Another asymptotic condition on the presence of 1-factors of a d-partite graph that is proved in [60] makes it possible to characterize matrices with positive permanent with the use of the number of ones in planes of arbitrary dimension:

Theorem 31. *Suppose that $1 \le k \le d-1$ and A is a d-dimensional $(0,1)$-matrix of order n. Choose a direction of k-dimensional planes and denote by N_k and N_{d-k} the minimal numbers of ones in the k-dimensional planes of this direction and the $(d-k)$-dimensional planes of the orthogonal direction respectively. If*

$$\frac{N_k}{n^k} + \frac{N_{d-k}}{n^{d-k}} \ge 1 + o(1) \quad \textit{as} \quad n \to \infty$$

then $\operatorname{per} A > 0$.

For searching for matrices with positive permanent, we can use not only d-partite hypergraphs but also arbitrary d-uniform hypergraphs. In particular, Theorem 2 implies that the permanent of the adjacency matrix of every hypergraph containing a 1-factor is always positive. Of course, not all multidimensional $(0,1)$-matrices can be the adjacency matrix of a hypergraph. Refer to a d-dimensional $(0,1)$-matrix of order n, where d divides n, as a *hypergraph* matrix if it is the adjacency matrix of some d-uniform hypergraph on n vertices.

Recently many works have been published that address the problem of the existence of 1-factors of uniform hypergraphs. Therefore, we expose only some applications of these articles to multidimensional matrices.

One of the first results on the existence of 1-factors of hypergraphs was obtained in [61]. For hypergraph matrices, it implies

Theorem 32. *If each hyperplane of a d-dimensional hypergraph matrix of order n contains at least $(1-1/d)\cdot(n-1)!/(n-d)!$ ones then its permanent is greater than zero.*

The definition of hypergraph matrix implies that not all of its planes contain 1s. Refer to a k-dimensional plane in a d-dimensional hypergraph matrix of order n as *essential* if it can contain 1s.

Denote by $t(d,n)$ the minimal number of ones in essential one-dimensional planes of a d-dimensional hypergraph matrix of order n sufficient for a corresponding hypergraph to have a 1-factor. In other words, if all essential one-dimensional planes of a hypergraph matrix contain more than $t(d,n)$ ones then its permanent is greater than zero. The following results were obtained for $t(d,n)$ in [62]:

Theorem 33. *For all $d \geq 3$ and sufficiently large n divisible by d, the number of ones in essential one-dimensional planes of d-dimensional hypergraph matrices of order n sufficient for the existence of a 1-factor is equal to*

$$t(d,n) = \begin{cases} n/2+3-d, & \text{if } d/2 \text{ is even and } n/d \text{ is odd}, \\ n/2+5/2-d, & \text{if } d \text{ is odd and } (n-1)/2 \text{ is odd}, \\ n/2+3/2-d, & \text{if } d/2 \text{ is even and } (n-1)/2 \text{ is even}, \\ n/2+2-d, & \text{otherwise}. \end{cases}$$

For hypergraph matrices, conditions for the positivity of the permanent can be formulated not only with the use of the number of 1s in the hyperplanes and essential one-dimensional planes but also with the help of the number of 1s in essential planes of other dimensions. Restating a result of [60], we obtain the following sufficient condition for the positivity of the permanent of hypergraph matrices, which is asymptotically exact:

Theorem 34. *Let A be a hypergraph d-dimensional matrix of order n. If for $k < d/2$ each essential k-dimensional plane A contains at least $(1/2+o(1)) \cdot n!/(n-k)!$ ones then the permanent of A is greater than zero.*

For $k \geq d/2$, a proper bound for the number of 1s in the essential k-dimensional planes of hypergraph matrices guaranteeing the positivity of the permanent is not obtained yet. In [63], Kühn and Osthus made the conjecture on what degree of k-vertex subsets of a hypergraph is sufficient for the existence of a 1-factor of the hypergraph. For hypergraph matrices, this conjecture can be reformulated as

Conjecture 13. *There exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all d-dimensional hypergraph matrices A of order n, if each essential k-dimensional plane A contains at least*

$$\max\left\{\frac{1}{2},\ 1-\left(\frac{d-1}{d}\right)^{d-k}\right\}\frac{(n-d+k)!}{(n-d)!}(1+o(1))$$

ones then the permanent of A is greater than zero.

In conclusion, let us give a simple criterion for positivity of the permanent of symmetric matrices. Refer to a d-dimensional matrix A of order n as *symmetric* if $a_\alpha = a_\beta$ for every indices α and β connected by a permutation of coordinates. In other words, a matrix is symmetric whenever it does not change under transposition. Note that the adjacency matrices of hypergraphs are multidimensional symmetric $(0,1)$-matrices for which the main diagonal of a two-dimensional plane is composed of zeros.

Given a matrix A, construct the set $N(A) = \{N_1,\dots,N_k\}$, where N_i ranges over all d-element multisets of elements $\{1,\dots,n\}$ corresponding to the indices of the unit entries of A. By the Hall's marriage theorem, we can prove

Proposition 14. *Let A be a symmetric d-dimensional matrix of order n. If, from the elements of $N(A)$, we can choose a family of sets $\{S_1,\dots,S_n\}$ such that each number $m \in \{1,\dots,n\}$ occurs in the multiset union of these sets exactly d times then the permanent of A is greater than zero.*

For d-uniform hypergraphs, from this assertion it is easy to deduce

Corollary 2. *Every d-regular d-uniform hypergraph has a regular orientation.*

8. THE MAIN PROBLEMS OF THE THEORY OF MULTIDIMENSIONAL PERMANENTS

1. Find sufficient conditions useful for applications for positivity of the permanent of k-stochastic matrices and arbitrary multidimensional (0,1)-matrices.

2. Study how the permanent changes under various transformations of multidimensional matrices (for example, in constructions of polystochastic matrices in Section 5).

3. Generalize Bregman theorem to the multidimensional case so that equality therein be attained at the widest possible class of matrices.

4. Describe the matrices at which the minimum and maximum of the permanent are attained for polystochastic matrices of even and odd dimensions.

5. Propose new constructions of multidimensional matrices whose permanent makes it possible to find the number of the already-known combinatorial objects.

6. Strengthen the available bounds for permanents of Section 2 expressing the number of various combinatorial objects.

7. Describe the vertices of the polytope of polystochastic and k-stochastic matrices and study the properties of their permanents.

8. Find an attainable bound for the number of 1-factors of a hypergraph in terms of the permanent of its adjacency matrix.

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